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The category of Fresnel kernels

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Abstract. *Symplectic relations and their generating functions have found extensive applications in classical mechanics. In the present paper we undertake the study of the correspondence between generating functions of symplectic relations and kernels of integral operators of quantum theories. As a first step we study this correspondence in the case of linear symplectic relations generated by quadratic functions. The theory is sufficiently complicated even in this simple case. Additional complications must be expected in the general nonlinear theory due to the fact that the composition of regular nonlinear symplectic relations is in general singular and that nonlinear symplectic relations in general do not have global generating functions. The present paper is a continuation of the study of linear symplectic relations undertaken in* [2].

1. LINEAR SYMPLECTIC RELATIONS

A *symplectic form* on a vector space P is a bilinear skew symmetric non-degenerate form on P.

A *symplectic space* is a pair (P, ω), where P a vector space and ω is a symplectic form on P.

Let (P, ω) be a symplectic space. If E is a subspace of P then the subspace

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$$\{p \in P \mid \omega(e,p) = 0 \ \text{ for each } \ e \in E\}$$

is denoted by $E^{\S}$. A subspace E of P is said to be *isotropic* if $E^{\S} \supset E$, *Lagrangian* if $E^{\S} = E$ and *coisotropic* if $E^{\S} \subset E$.

For two symplectic spaces (P_1, ω_1) and (P_2, ω_2) we denote by $\omega_1 \oplus \omega_2$ the symplectic form on $P_1 \oplus P_2$ defined by $(\omega_1 \oplus \omega_2)(p_1, p_2) = \omega_1(p_1) + \omega_2(p_2)$ for $p_1 \in P_1, p_2 \in P_2$. The symplectic space $(P_1 \oplus P_2, \omega_1 \oplus \omega_2)$ is called the *direct sum* of spaces (P_1, ω_1) and (P_2, ω_2).

a) *The category of linear symplectic relations.*

A *linear symplectic relation* is a triple $\rho = ((P_2, \omega_2), (P_1, \omega_1), R)$ where (P_1, ω_1) and (P_2, ω_2) are symplectic spaces and R is a Lagrangian subspace of $(P_2, \omega_2) \oplus \oplus (P_1, -\omega_1)$. In this case we say that ρ is a linear symplectic relation from (P_1, ω_1) to (P_2, ω_2) and we denote this by

$$\rho : (P_1, \omega_1) \longrightarrow (P_2, \omega_2).$$

R is called the *graph* of ρ (we also write $R = \text{Graph}\ \rho$).

Symplectic spaces with linear symplectic relations form a category. We denote this category by *LSR*. The composition in *LSR* is the usual composition of relations. The identity morphism from (P, ω) to (P, ω) is given by the identity relation:

$$1_{(P,\omega)} = ((P,\omega), (P,\omega), R) \ \text{ with } \ R = \{p \oplus p' \in P \oplus P \mid p' = p\}.$$

One can show that if $\rho = ((P_2, \omega_2), (P_1, \omega_1), R)$ is a linear symplectic relation and E is a subspace of P_1 then

$$\rho(E^{\S}) = \rho(E)^{\S},$$

where $\rho(E) = \{p_2 \in P_2 \mid \text{there exists } e \in E \text{ such that } p_2 \oplus e \in R\}$ is the *image* of E under ρ. Hence for E isotropic (resp. Lagrangian, coisotropic), $\rho(E)$ is also isotropic (resp. Lagrangian, coisotropic).

b) *The functor* «†». For each linear symplectic relation $\rho = ((P_2, \omega_2), (P_1, \omega_1), R)$ we introduce the *adjoint relation* $\rho^{\dagger} = ((P_1, \omega_1), (P_2, \omega_2), R^{\dagger})$, where

$$R^{\dagger} = \{p_1 \oplus p_2 \in P_1 \oplus P_2 \mid p_2 \oplus p_1 \in R\}.$$

The map

$$(P, \omega) \longmapsto (P, \omega)^{\dagger} = (P, \omega), \ \ \rho \longmapsto \rho^{\dagger}$$

defines a contravariant functor in *LSR*. We have $(\rho^{\dagger})^{\dagger} = \rho$ for each relation ρ.

c) *Special morphisms.* A linear symplectic relation $\rho : (P_1, \omega_1) \to (P_2, \omega_2)$ is said to be

(i) an *epimorphism*, if for any $\alpha : (P_2, \omega_2) \to (P, \omega)$, $\beta : (P_2, \omega_2) \to (P, \omega)$

$$\alpha \circ \rho = \beta \circ \rho \text{ implies } \alpha = \beta$$

(ii) a *retraction*, if there exists $\sigma : (P_2, \omega_2) \to (P_1, \omega_1)$ such that $\rho \circ \sigma = 1_{(P_2, \omega_2)}$

($\mathrm{i}^\dagger$) a *monomorphism*, if for $\alpha, \beta : (P, \omega) \to (P_1, \omega_1)$

$$\rho \circ \alpha = \rho \circ \beta \text{ implies } \alpha = \beta$$

($\mathrm{ii}^\dagger$) a *coretraction*, if there exists $\sigma : (P_2, \omega_2) \to (P_1, \omega_1)$ such that $\sigma \circ \rho = 1_{(P_1, \omega_1)}$

(—) an *isomorphism*, if it is at the same time a retraction and a coretraction.

In *LSR* conditions (i) and (ii) are equivalent. They are also equivalent to each of the following conditions:

(iii) $\rho \circ \rho^\dagger = 1_{(P_2, \omega_2)}$

(iv) $\rho(P_1) = P_2$.

Similarly ($\mathrm{i}^\dagger$) and ($\mathrm{ii}^\dagger$) are both equivalent to

($\mathrm{iii}^\dagger$) $\rho^\dagger \circ \rho = 1_{(P_1, \omega_1)}$,

or

($\mathrm{iv}^\dagger$) $\rho^\dagger(P_2) = P_1$.

The functor «†» changes epimorphisms into monomorphisms and vice versa.

For any isomorphism ρ we have $\rho^{-1} = \rho^\dagger$. Symplectic spaces with isomorphisms form a subcategory of *LSR*. The set of all monomorphisms and the set of all epimorphisms provide two other examples of subcategories.

d) *Reductions*. For any coisotropic subspace K in a symplectic space (P, ω) the *reduced symplectic space* of (P, ω) *with respect to* K is a symplectic space $(P_{[K]}, \omega_{[K]})$. The space $P_{[K]}$ is the quotient space $K/K^\S$ and the form $\omega_{[K]}$ is defined by

$$\omega_{[K]}(\bar{p}_1, \bar{p}_2) = \omega(p_1, p_2),$$

where $\bar{p}_1, \bar{p}_2 \in P_{[K]}$ and $p_1, p_2 \in P$ are such that $p_1 \in \bar{p}_1$ and $p_2 \in \bar{p}_2$. The triple $r_{[K]} = ((P_{[K]}, \omega_{[K]}), (P, \omega), R)$ where

$$R = \{\bar{p} \oplus p \in P_{[K]} \oplus P \mid p \in \bar{p}\},$$

is a linear symplectic relation called the *reduction relation* associated with K. This relation is an epimorphism.

The following decomposition theorem holds:

If $\rho : (P_1, \omega_1) \to (P_2, \omega_2)$ is a linear symplectic relation, then there exists a unique isomorphsim ρ_0 such that

$$\rho = r_{[K_2]} \circ \rho_0 \circ r_{[K_1]},$$

where $K_1 = \rho^\dagger(P_2)$, $K_2 = \rho(P_1)$.

e) *Selfadjoint idempotents.* A linear symplectic relation $\chi : (P, \omega) \to (P, \omega)$ is called *selfadjoint* if $\chi^{\dagger} = \chi$. The following proposition is a simple consequence of the decomposition theorem.

PROPOSITION 1. *If* $\chi : (P, \omega) \to (P, \omega)$ *is selfadjoint, then the isomorphism* χ_0 *appearing in the decomposition*

$$\chi = r_{[K]}^{\dagger} \circ \chi_0 \circ r_{[K]}, \; K = \chi(P)$$

satisfies $\chi_0 \circ \chi_0 = 1_{(P_{[K]}, \, \omega_{[K]})}$.

The next proposition characterizes selfadjoint idempotents.

PROPOSITION 2. *Let* $\delta : (P, \omega) \to (P, \omega)$ *be a linear symplectic relation and let* $K = \delta(P)$. *The following conditions are equivalent:*

(i) $\delta^{\dagger} = \delta$ *and* $\delta \circ \delta = \delta$

(ii) $\delta = r_{[K]}^{\dagger} \circ r_{[K]}$

(iii) *there exists* $\rho : (P, \omega) \to (P_1, \omega_1)$ *such that* $\delta = \rho^{\dagger} \circ \rho$.

Proof. (i) $\Rightarrow$ (ii). Let $r \doteq r_{[K]}$. From Proposition 1 it follows that

$$\delta = r^{\dagger} \circ \delta_0 \circ r, \quad \text{where} \quad \delta_0 \circ \delta_0 = 1_{(P_{[K]}, \omega_{[K]})}.$$

Hence, $\delta = \delta \circ \delta = r^{\dagger} \circ \delta_0 \circ r \circ r^{\dagger} \circ \delta_0 \circ r = r^{\dagger} \circ \delta_0 \circ \delta_0 \circ r = r^{\dagger} \circ r$.

(ii) $\Rightarrow$ (i). For $r = r_{[K]}$ and $\delta = r^{\dagger} \circ r$, we have

$$\delta^{\dagger} = (r^{\dagger} \circ r)^{\dagger} = r^{\dagger} \circ r = \delta$$

and

$$\delta \circ \delta = (r^{\dagger} \circ r) \circ (r^{\dagger} \circ r) = r^{\dagger} \circ (r \circ r^{\dagger}) \circ r = r^{\dagger} \circ r = \delta.$$

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii). Let $K' = \rho^{\dagger}(P_1)$, $r = r_{[K']}$. Then $\rho = \mu \circ r$, where μ is a monomorphism, hence $\delta = r^{\dagger} \circ \mu^{\dagger} \circ \mu \circ r = r^{\dagger} \circ r$ and $K' = K$. ■

DEFINITION 1. A linear symplectic relation $\delta : (P, \omega) \to (P, \omega)$ is called *positive*, if here exists a linear symplectic relation ρ such that $\delta = \rho^{\dagger} \circ \rho$.

It follows from Proposition 2 that a relation ρ is positive if and only if it is selfadjoint and idempotent. It follows also that positive symplectic relations are in one-to-one correspondence with coisotropic subspaces.

2. QUADRATIC FUNCTIONS

We denote by V^* the dual space of a vector space V. For $w \in V^*$ and $v \in V$ the value $w(v)$ of w on v is denoted also by $\langle v, w \rangle$. If X is a subspace of V then $X^0 = \{w \in V^* \mid \langle x, w \rangle = 0 \text{ for } x \in X\}$ is called the *annihilator* of X (in V^*). The identity mapping of V onto itself will be denoted by ι_V. The mapping $\iota_V \oplus (-\iota_{V^*}) : V \oplus V^* \to V \oplus V^*$ will be denoted by τ_V.

a) *The phase functor*

A *linear relation* is a triple $\alpha = (Y, X, A)$, where X and Y are vector spaces and A is a subspace of $Y \oplus X$. Vector spaces with linear relations form a category, denoted by LR. The composition in LR is the usual composition of relations. The identity morphism from V to V is given by ι_V.

Let V be a vector space. The *phase space* of V is the symplectic space $PhV = (V \oplus V^*, \omega_V)$, where ω_V is the canonical symplectic form on $V \oplus V^*$:

$$\omega_V(v \oplus w, v_1 \oplus w_1) = w(v_1) - w_1(v) = \langle v_1, w \rangle - \langle v, w_1 \rangle.$$

If X and Y are two vector spaces then the symplectic space $PhX \oplus PhY$ can be canonically identified with the phase space $Ph(X \oplus Y)$. We shall use this identification in the sequel.

To each linear relation $\alpha = (Y, X, A)$ there corresponds the linear symplectic relation $Ph\alpha = (PhY, PhX, R)$, where

$$R = (\iota_{Y \oplus Y^*} \oplus \tau_X)(A \oplus A^0).$$

Here $A \oplus A^0 \subset Ph(Y \oplus X) = PhY \oplus PhX$.

The map

$$V \longmapsto PhV, \quad \alpha \longmapsto Ph\alpha$$

is a covariant functor from LR to LSR. It is called the *phase functor*.

b) *Phase spaces and generating functions of Lagrangian subspaces*

Let L be a Lagrangian subspace of a phase space PhV and let $C = \pi_V(L)$, where π_V is the projection of $V \oplus V^*$ onto V. The *generating function* of L is the quadratic function Q on C defined by

$$Q(v) = \frac{1}{2} \langle v, w \rangle,$$

where $v \in C$ and w is an element of V^* such that $v \oplus w \in L$.

Conversely, a quadratic function Q on a subspace C of V *generates* a Lagrangian subspace L of PhV as follows:

$$L = \{v \oplus w \in V \oplus V^* \mid v \in C \quad \text{and} \quad \langle x, w\rangle = \langle x, \mathrm{d}Q(v)\rangle \quad \text{for each} \quad x \in C\}.$$

Here $\mathrm{d}Q$ denotes the mapping from C to C^* defined by

$$\langle x, \mathrm{d}Q(v)\rangle = Q(v + x) - Q(v) - Q(x).$$

The above formulae establish a bijective correspondence hetween Lagrangian subspaces of $Ph\,V$ and quadratic functions on subspaces of V.

If $C = V$ then both L and Q are said to be *regular*.

c) *Linear phase relations and their generating functions*

A *linear phase relation* is a linear symplectic relation from one phase space to another. Phase spaces with linear phase relations form a category which we denote by LPR. This is the full subcategory of LSR whose class of objects consists of all phase spaces:

$$Ob(LPR) = Ph(Ob(LR)).$$

If $\rho = (PhY, PhX, R)$ is a linear phase relation then $R' = (\iota_{(Y \oplus Y^*)} \oplus \tau_X)(R)$ is a Lagrangian subspace of $Ph\,Y \oplus PhX = Ph(Y \oplus X)$. The generating function of R' will be called the *generating function* of ρ and will be denoted by Q_ρ. This function is defined on the subspace $C_\rho = \pi_{Y \oplus X}(R)$. The adjoint relation is generated by the function $Q_{\rho^\dagger} : C_{\rho^\dagger} \to \mathbb{R}$, where

$$C_{\rho^\dagger} = \{x \oplus y \in X \oplus Y \mid y \oplus x \in C\}$$

and

$$Q_{\rho^\dagger}(x \oplus y) = -Q_\rho(y \oplus x).$$

Note that $C_{Ph\alpha} = \text{Graph}\,(\alpha)$ and $Q_{Ph\alpha} = 0$ for each linear relation α.

d) *The category of quadratic functions*

We now introduce a new category, called the category of quadratic functions and denoted by QF. This category is isomorphic to LPR. It is obtained from LPR by replacing linear phase relations by their generating functions. The formal definition of QF is the following:

(i) objects of QF are vector spaces,

(ii) morphisms from a vector space X to a vector space Y are quadratic functions defined on subspaces of $Y \oplus X$. The set of all such functions will be denoted by $QF(Y, X)$,

(iii) the composition of two morphisms $Q_2 \in QF\,(Z, Y)$ and $Q_1 \in QF(Y, X)$ will be denoted by $St_Y(Q_2 + Q_1)$. It is defined by the following variational principle. Let Q_1 and Q_2 be defined on $C_1 \subset Y \oplus X$ and $C_2 \subset Z \oplus Y$, respectively. For $z \oplus x \in Z \oplus X$ let $Y_{z \oplus x} = \{y \in Y \mid z \oplus y \in C_2,\ y \oplus x \in C_1\}$ and let the function

$$Q_{z\oplus x}(y) = Q_2(z \oplus y) + Q_1(y \oplus x).$$

Then $St_Y(Q_2 + Q_1)$ is defined on

$$C = \{z \oplus x \in Z \oplus X \mid Y_{z\oplus x} \neq \phi \text{ and } Q_{z\oplus x} \text{ has a stationary point}\}$$

by

$$St_Y(Q_2 + Q_1)(z \oplus x) = Q_{z\oplus x}(y),$$

where y is a stationary point (critical point) of $Q_{z\oplus x}$.

The symbol St_Y refers to the fact that $St_Y(Q_1 + Q_1)$ is defined as the collection of stationary values of the functions $Q_{z\oplus x}$.

e) *Morphisms with zero-dimensional domain*

Each Lagrangian subspace L of a symplectic space (P, ω) defines the linear symplectic relation $\rho_L : \{0\} \to (P, \omega)$, given by the formula

$$\text{Graph}\, \rho_L = L \oplus \{0\}.$$

Conversely, if $\rho : \{0\} \to (P, \omega)$, then $\rho = \rho_L$ for $L = \rho(\{0\})$. If $(P, \omega) = PhV$, then ρ_L is a linear phase relation and the generating function of L coincides with the generating function of ρ_L (if we identify V and $V \oplus \{0\}$). In other words, quadratic functions on subspaces of V can be identified with the elements of $QF(V, \{0\})$.

Now let K be a Lagrangian subspace of $Ph\Lambda$ and $\rho : Ph\Lambda \to PhV$. If $L = \rho(K)$ then $\rho_L = \rho \circ \rho_K$. Combining this with the composition rules in QF we obtain the following formula for the generating function of L:

$$Q_L = St_\Lambda(Q_\rho + Q_K). \tag{1}$$

f) *Morse families*

Let V and Λ be two vector spaces. Let M be a quadratic function on $V \oplus \Lambda$. M is uniquely represented in the following form

$$M(v, \lambda) = Q_0(v) + \frac{1}{2}\langle \lambda, A\lambda \rangle - \langle \lambda, Bv \rangle, \tag{2}$$

where $v \in V$, $\lambda \in \Lambda$, Q_0 is a quadratic function on V, $A : \Lambda \to \Lambda^*$, $A^* = A$, and $B : V \to \Lambda^*$. Let $\rho : Ph\Lambda \to PhV$ be the linear phase relation generated by the quadratic function M (i.e. $M = Q_\rho$).

Translations parallel to Λ and leaving M unchanged form a subspace of Λ, denoted by Λ_0. In other words,

$$\Lambda_0 = \{\lambda_0 \in \Lambda \mid M(v, \lambda + \lambda_0) = M(v, \lambda) \text{ for each } v \in V, \lambda \in \Lambda\}.$$

PROPOSITION 3. *The following conditions are equivalent*

(i) $\Lambda_0 = \{0\}$

(ii) $(\ker \mathrm{d}M) \cap (\{0\} \oplus \Lambda) = \{0\}$

(iii) $\mathrm{Im}\, A + \mathrm{Im}\, B = \Lambda^*$ ($\mathrm{Im}\, A$ *denotes the image of* A)

(iv) $\rho^\dagger(\{0\}) \cap (\Lambda \oplus \{0\}) = \{0\}$.

Proof. (i) $\Longleftrightarrow$ (ii) $\Lambda_0 = \ker \mathrm{d}M|_{\{0\}\oplus\Lambda}$

(ii) $\Longleftrightarrow$ (iii) $\mathrm{d}M|_{\{0\}\oplus\Lambda} = A + (-B^*)$ is injective if and only if $[A + (-B^*)]^* = A + (-B)$ is surjective

(ii) $\Longleftrightarrow$ (iv) follows from the equality

$$\rho^\dagger(\{0\}) \cap (\Lambda \oplus \{0\}) = (\ker \mathrm{d}M|_{\{0\}\oplus\Lambda}) \oplus \{0\}. \qquad \blacksquare$$

If the conditions of Proposition 3 are satisfied, we say that M is a *Morse family* of functions on Λ *indexed over* V. The Lagrangian subspace $L = \rho(\Lambda \oplus \{0\})$ of $Ph\, V$ is said to be *generated* by the Morse family M. By formula (1), the generating function of L is given by

$$Q = St_\Lambda(M + 0).$$

The conditions listed in Proposition 3 mean that for each $v \in C = \pi_V(L)$ there is exactly one stationary point.

PROPOSITION 4. *Let* $\rho : Ph\,\Lambda \to Ph\, V$ *be the linear phase relation generated by a quadratic function* M *on* $V \oplus \Lambda$. *Let* M *be given by formula* (2). *Then the generating function* Q *of* $L = \rho(\Lambda \oplus \{0\})$ *is defined on* $C = B^{-1}(\mathrm{Im}\, A)$, *and for* $v \in C$,

$$Q(v) = -\frac{1}{2}\langle \lambda, Bv\rangle + Q_0(v),$$

where λ *is any element of* Λ *such that* $A\lambda = Bv$.

Proof. $v \oplus w \oplus \lambda \oplus \varkappa \in \mathrm{Graph}\, \rho$ if and only if

$$-\varkappa = A\lambda - Bv \quad \text{and} \quad w = \mathrm{d}Q_0(v) - B^*\lambda.$$

Hence $v \oplus w \in \rho\,(\Lambda \oplus \{0\})$ if and only if there exists $\lambda \in \Lambda$ such that

$$A\lambda = Bv \quad \text{and} \quad w = \mathrm{d}Q_0(v) - B^*\lambda.$$

The condition for v is therefore

$$Bv \in \mathrm{Im}\, A, \quad \text{or} \quad v \in B^{-1}(\mathrm{Im}\, A).$$

For $v \in C$ we have

$$Q(v) = \frac{1}{2}\langle v, w\rangle = \frac{1}{2}\langle v, \mathrm{d}Q_0(v)\rangle - \frac{1}{2}\langle v, B^*\lambda\rangle = Q_0(v) - \frac{1}{2}\langle \lambda, Bv\rangle. \quad \blacksquare$$

Morse families provide an alternative representation of Lagrangian subspaces by quadratic functions.

PROPOSITION 5. *Each Lagrangian subspace L of PhV is generated by a Morse family indexed over V.*

Proof. Let Q be the generating function of L defined on a subspace C of V. We set $\Lambda = C^0$ and define

$$M(v, \lambda) = -\langle v, \lambda\rangle + \widetilde{Q}(v)$$

where $v \in V$, $\lambda \in \Lambda$, and $\widetilde{Q}$ is any extension of Q to the space V. From Proposition 4 we see that M is a Morse familly generating L. ■

Morse families can be used to generate linear phase relations. A linear phase relation $\rho : PhX \longmapsto PhY$ is said to be *generated* by a Morse family $M : (Y \oplus X) \oplus \Lambda \to \mathbb{R}$ if $St_\Lambda(M) = Q_\rho$.

3. SCHWARTZ SPACES AND INTEGRATION OF DISTRIBUTIONS

a) *Densities on a vector space*

Let V be a vector space of dimension n. For any real non negative σ we denote by $|V^*|^\sigma$ the complex one-dimensional space of mappings $\mu : \overset{n}{\Lambda} V \to \mathbb{C}$ such that $\mu(tw) = |t|^\sigma \mu(w)$ for $t \in \mathbb{R}$, $w \in \overset{n}{\Lambda} V$ ($0^0 = 1$ is assumed). $|V^*|^1$ is denoted also by $|V^*|$. The *product* $\mu_1 \cdot \mu_2$ of $\mu_1 \in |V^*|^{\sigma_1}$ and $\mu_2 \in |V^*|^{\sigma_2}$, defined by

$$(\mu_1 \cdot \mu_2)(w) = \mu_1(w)\,\mu_2(w)$$

belongs to $|V^*|^{\sigma_1+\sigma_2}$. We can also divide $\mu_1 \in |V^*|^{\sigma_1}$ by $\mu_2 \in |V^*|^{\sigma_2}$ if $\sigma_2 \leqslant \sigma_1$ and $\mu_2 \neq 0$. The *quotient* $\frac{\mu_1}{\mu_2}$ defined by

$$\frac{\mu_1}{\mu_2}(w) = \frac{\mu_1(w)}{\mu_2(w)}$$

for non-zero $w \in \overset{n}{\Lambda} V$, belongs to $|V^*|^{\sigma_1-\sigma_2}$.

An element μ of $|V^*|^\sigma$ is said to be *positive* if $\mu(w)$ is positive for each $w \neq 0$.

Positive elements of $|V^*|^\sigma$ can be raised to non-negative powers. The *power* μ^γ is defined by

$$\mu^\gamma(w) = \mu(w)^\gamma,$$

and belongs to $|V^*|^{\sigma\gamma}$.

Any isomorphism $A : V \to W$ of two vector spaces gives rise to an isomorphism $A : |V^*|^\sigma \to |W^*|^\sigma$ (denoted by the same letter).

If $V = X \oplus Y$ then $|V^*|^\sigma = |X^*|^\sigma \otimes |Y^*|^\sigma$. Here the tensor product $\mu \otimes \nu$ of $\mu \in |X^*|^\sigma$ and $\nu \in |Y^*|^\sigma$ is identified with an element of $|V^*|^\sigma$ as follows:

$$(\mu \otimes \nu)(\overline{x} \wedge \overline{y}) = \mu(\overline{x})\nu(\overline{y}), \quad \overline{x} \in \overset{m}{\wedge} X,\ \overline{y} \in \overset{n-m}{\wedge} Y,\ m = \dim X.$$

We also have a canonical isomorphism

$$|V^*|^\sigma = |X^*|^\sigma \otimes |(V/X)^*|^\sigma.$$

This isomorphism is defined by

$$|V^*|^\sigma = |X^*|^\sigma \otimes |Y^*|^\sigma = |X^*|^\sigma \otimes |(V/X)^*|^\sigma,$$

where Y is an arbitrary complement of X in V.

A σ-*density* on V is a mapping from V to $|V^*|$.

1 - densities are called *densities* and $\frac{1}{2}$ - densities are called *half-densities*. 0 - densities are simply complex functions on V.

We shall often identify elements of $|V^*|$ with constant σ - densities. There is a constant positive density associated with each linear coordinate system $(v^1, \ldots, v^n)$ of V. This density denoted by $\mathrm{d}v^1 \ldots \mathrm{d}v^n$ is characterized by

$$\mathrm{d}v^1 \ldots \mathrm{d}v^n \left(\frac{\partial}{\partial v^1} \wedge \ldots \wedge \frac{\partial}{\partial v^n} \right) = 1.$$

With respect to a coordinate system $(v^1, \ldots, v^n)$ a σ-density ζ is represented by a function f such that

$$\zeta = f \cdot (\mathrm{d}v^1 \ldots \mathrm{d}v^n)^\sigma.$$

Using a different coordinate system $(\tilde{v}^1, \ldots, \tilde{v}^n)$ we obtain for the same ζ a function $\tilde{f}$ such that

$$\tilde{f} = f \left| \frac{\partial(v^1, \ldots, v^n)}{\partial(\tilde{v}^1, \ldots, \tilde{v}^n)} \right|^\sigma.$$

Densities on a vector space V can be integrated. The integral of a density ζ is calculated from the formula

$$\int_V \zeta = \int_{\mathbb{R}^n} f(v^1, \ldots, v^n)\, dv^1 \ldots dv^n$$

in any coordinate system $(v^1, \ldots, v^n)$ in V.

The *modulus* of a form $\Omega \in \overset{n}{\wedge} V^*$ is the density $|\Omega|$ defined by

$$|\Omega|(w) = |\Omega(w)| \quad \text{for} \quad w \in \overset{n}{\wedge} V.$$

b) *Schwartz spaces S and S' associated with a vector space*

Let V be a vector space. By $S(V)$ (the *Schwartz space* of V) we denote the space of smooth, rapidly vanishing at infinity half-densities on V. More exactly, a half-density ψ on V belongs to $S(V)$ if

(i) $\psi : V \to |V^*|^{1/2}$ is a C^∞-mapping,

(ii) for any differential operator A with polynomial coefficients

$$\|\psi\|_A = \sup_{v \in V} |A\psi(v)| < \infty.$$

Here $|\cdot|$ denotes a fixed norm on $|V^*|^{1/2}$.

The space $S(V)$ is endowed with natural topology induced by seminorms $\|\cdot\|_A$. If we choose a coordinate system $(v^1, \ldots, v^n)$ on V, then the same topology can be described as induced by the following family of norms:

$$\|\psi\|^\alpha_\beta = \sup_{v \in V} |v^\alpha D_\beta \psi(v)|, \tag{3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indices and $v^\alpha D_\beta$ denotes the differential operator

$$\prod_{k=1}^n (v^k)^{\alpha_k} \cdot \prod_{l=1}^n \left(\frac{\partial}{\partial v^1}\right)^{\beta_i}.$$

We denote by $S'(V)$ the dual space of $S(V)$:

$$S'(V) = S(V)'.$$

If $\mathscr{K} \in S'(V)$ and $\psi \in S(V)$ then the value of $\mathscr{K}$ on ψ is denoted by $\langle \psi, \mathscr{K} \rangle$. $S'(V)$ is endowed with the weak topology, i.e. the one induced by mappings $\mathscr{K} \longmapsto \langle \psi, \mathscr{K} \rangle$ for $\psi \in S(V)$.

An element ψ of $S(V)$ may be treated as a functional on $S(V)$ as follows:

(4) $$S(V) \ni \phi \longmapsto \langle \phi, \psi \rangle = \int_V \phi \cdot \psi.$$

This defines a continuous inclusion $S(V) \subset S'(V)$.

Elements of $S'(V)$ are called *generalized half-densities* on V. The *scalar product* $\langle \phi \mid \psi \rangle$ of $\phi, \psi \in S(V)$ is defined by

(5) $$\langle \phi \mid \psi \rangle = \langle \overline{\phi}, \psi \rangle = \int_V \overline{\phi} \cdot \psi,$$

where $\overline{\phi}$ is the complex conjugate of ϕ. Note that $\langle \phi \mid \psi \rangle = \langle \overline{\phi}, \psi \rangle$ is defined also for $\psi \in S'(V)$.

If X and Y are two vector spaces, then

$$S(X \oplus Y) = S(X) \mathbin{\overline{\otimes}} S(Y)$$

and

$$S'(X \oplus Y) = S'(X) \mathbin{\hat{\otimes}} S'(Y),$$

where $-$ and $\frown$ denote suitable completions of algebraic tensor products (see [3]).

Each element $\mathscr{K}$ of $S'(X \oplus Y)$ defines a continuous linear mapping $\mathscr{K}_Y^X : S(X) \to S'(Y)$ by

$$\langle \phi, \mathscr{K}_Y^X \psi \rangle = \langle \psi \otimes \phi, \mathscr{K} \rangle$$

for $\psi \in S(X)$, $\phi \in S(Y)$. It follows from the kernel theorem that all continuous linear mappings from $S(X)$ to $S'(Y)$ are obtained in this way.

Together with $\mathscr{K}_Y^X$ we have $\mathscr{K}_X^Y : S(Y) \to S'(X)$, defined by

(6) $$\langle \psi, \mathscr{K}_X^Y \phi \rangle = \langle \psi \otimes \phi, \mathscr{K} \rangle$$

for $\psi \in S(X)$, $\phi \in S(Y)$.

c) *Integration conventions*

If $\phi, \psi \in S(V)$ then the density $\phi \cdot \psi$ is Lebesgue integrable. The integral in formula (4) is the ordinary Lebesgue integral. We intend to use the integration symbol to denote conditionally convergent improper integrals and also to denote operations which are integrals only formally in a generalized sense defined below. The generalization in based on the observation that for a Lebesgue integrable density ζ on V

$$\int_V \zeta = \hat{\zeta}(0),$$

where $\hat{\zeta}$ is the Fourier transform of ζ:

(7) $$\hat{\zeta}(w) = \int_V e^{-i\langle v,w\rangle}\zeta(v) \quad \text{for} \quad w \in V^*.$$

We first introduce auxiliary spaces associated with a vector space V:

(8) $$S(V)_1 = \mu \cdot S(V) = \{\mu\psi \mid \psi \in S(V)\} \qquad \textit{(Schwartz densities)}$$

and

(9) $$S(V)_0 = \frac{S(V)}{\mu} = \left\{\frac{\psi}{\mu} \mid \psi \in S(V)\right\} \qquad \textit{(Schwartz functions)},$$

where $\mu \in |V^*|^{1/2}$, $\mu \neq 0$ (any $\mu \neq 0$ gives the same result).

Note that the multiplication and the division by μ is also defined for $\mathscr{K} \in S'(V)$:

$$S'(V) \ni \mathscr{K} \longmapsto \mu \cdot \mathscr{K} \in [S(V)_0]', \text{ where } \langle f, \mu \cdot \mathscr{K}\rangle = \langle f \cdot \mu, \mathscr{K}\rangle \text{ for } f \in S(V)_0;$$

$$S'(V) \ni \mathscr{K} \longmapsto \frac{\mathscr{K}}{\mu} \in [S(V)_1]', \text{ where } \left\langle \zeta, \frac{\mathscr{K}}{\mu}\right\rangle = \left\langle \frac{\zeta}{\mu}, \mathscr{K}\right\rangle \text{ for } \zeta \in S(V)_1.$$

In analogy with (8) and (9) the images of these mappings are denoted by $S'(V)_1$ and $S'(V)_0$, respectively. It is easy to see that

$$[S(V)_0]' = S'(V)_1 \qquad \textit{(generalized densities)}$$

and

$$[S(V)_1]' = S'(V)_0 \qquad \textit{(generalized functions)}.$$

The Fourier transform (7) defines an isomorphism of $S(V)_1$ and $S(V)_0$. This isomorphism can be extended to $S'(V)_1$ in the usual way:

$$\langle \zeta, \hat{\mathscr{T}}\rangle = \langle \hat{\zeta}, \mathscr{T}\rangle$$

for $\zeta \in S(V^*)_1$, $\mathscr{T} \in S'(V)_1$. The Fourier transform of a generalized function $T \in S'(V)_0$ is by definition such $\mathscr{T} \in S'(V^*)_1$ that $\hat{\mathscr{T}} = \check{T}$, where $\check{T}$ is T composed with the reflection of V:

$$\langle \zeta, \check{T}\rangle = \langle \check{\zeta}, T\rangle \text{ for } \zeta \in S(V)_1;\ \check{\zeta}(v) = \zeta(-v).$$

We give here several useful examples of Fourier transforms.

EXAMPLE 1. *Special densities*. For each $\mu \in |V^*|$, $\mu \neq 0$ we denote by μ^* the unique element of $|(V^*)^*|$ such that

$$\mu \otimes \mu^* = \frac{1}{n!} \left| \left(\frac{\omega_V}{2\pi} \right)^n \right|, \quad n = \dim V.$$

Let δ_0 denote the Dirac density:

$$\langle f, \delta_0 \rangle = f(0) \quad \text{for} \quad f \in S(V)_0.$$

Then $\hat{\delta}_0 = 1$ and for each constant density $\mu \in |V^*|$, $\mu \neq 0$, we have

$$\hat{\mu} = \delta_0 / \mu^*.$$

EXAMPLE 2. If $f \in S(V)_0$ then

$$\hat{f} = \mu^* (\mu \cdot f)\hat{\ },$$

where μ is an arbitrary element of $|V^*|$ such that $\mu \neq 0$. The equality is valid also for f replaced by $T \in S'(V)_0$.

EXAMPLE 3. Let X be a subspace of V. If $\mathscr{Y} \in S'(X)_1$, we denote by $\mathscr{Y}\delta_X$ the element of $S'(V)_1$ defined by

$$\langle f, \mathscr{Y}\delta_X \rangle = \langle f|_X, \mathscr{Y} \rangle \quad \text{for} \quad f \in S(V)_0.$$

If $T \in S'(V/X)_0$, we denote by $T \otimes 1_X$ an element of $S'(V)_0$ defined by

$$\langle \zeta, T \otimes 1_X \rangle = \left\langle \int_X \zeta, T \right\rangle \quad \text{for} \quad \zeta \in S(V)_1.$$

We shall show that for $\mathscr{Y} \in S'(X)_1$ we have

$$(\mathscr{Y}\delta_X)\hat{\ } = \hat{\mathscr{Y}} \otimes 1_{X^0}.$$

Indeed, for $\zeta \in S(V^*)_1$ we have

$$\langle \zeta, (\mathscr{Y}\delta_X)\hat{\ } \rangle = \langle \hat{\zeta}, \mathscr{Y}\delta_X \rangle = \langle \hat{\zeta}|_X, \mathscr{Y} \rangle =$$

$$= \left\langle \left(\int_{X^0} \zeta \right)\hat{\ }, \mathscr{Y} \right\rangle = \left\langle \int_{X^0} \zeta, \hat{\mathscr{Y}} \right\rangle = \langle \zeta, \hat{\mathscr{Y}} \otimes 1_{X^0} \rangle.$$

DEFINITION 2. A generalized density $\mathscr{Y} \in S'(V)_1$ is said to be *integrable* if $\hat{\mathscr{Y}}$ is a continuous, slowly increasing (i.e. polynomially bounded) function. In this case the *integral* $\int_V \mathscr{Y}$ is defined by

$$\int_V \mathscr{Y} = \hat{\mathscr{Y}}(0).$$

Remark. L. Schwartz defined summable distributions in a different way (see [4]). One can show that the two definitions are equivalent.

EXAMPLE 4. The product of $\psi \in S(V)$ and $\mathscr{K} \in S'(V)$ is defined in the standard way:

$$\langle f, \psi \cdot \mathscr{K} \rangle = \langle f\psi, \mathscr{K} \rangle \quad \text{for } f \in S(V)_0.$$

We shall show that $\psi \cdot \mathscr{K}$ is integrable and $\int_V \psi \cdot \mathscr{K} = \langle \psi, \mathscr{K} \rangle$. For $\zeta \in S(V^*)_1$ we have

$$\langle \zeta, (\psi \cdot \mathscr{K})\hat{}\, \rangle = \langle \hat{\zeta}, \psi \cdot \mathscr{K} \rangle = \langle \psi \cdot \hat{\zeta}, \mathscr{K} \rangle =$$

$$= \left\langle \psi \cdot \int_{V^*} \zeta(w)\overline{\chi}_w, \mathscr{K} \right\rangle = \langle \overline{\epsilon} \cdot (\psi \otimes \zeta), \mathscr{K} \otimes 1 \rangle =$$

$$= \int_{V^*} \zeta(w) \langle \overline{\chi}_w \psi, \mathscr{K} \rangle,$$

where $\chi_w(v) = \epsilon(v, w) = e^{i\langle v, w \rangle}$. It turns out that

$$(10) \qquad (\psi \cdot \mathscr{K})\hat{}\, : V^* \longrightarrow \mathbf{C}; \quad w \longmapsto \langle \overline{\chi}_w \psi, \mathscr{K} \rangle$$

is a continuous function. By the continuity of $\mathscr{K}$ there exists a constant C such that

$$(11) \qquad |\langle \overline{\chi}_w \psi, \mathscr{K} \rangle| \leqslant C \sum_{\alpha, \beta} \| \overline{\chi}_w \psi \|_\beta^\alpha \quad ,$$

where $\| \cdot \|_\beta^\alpha$ are the norms from formula (3) and the sum is finite. It is easy to see that this sum is bounded by a polynomial.

EXAMPLE 5. *Fresnel integrals.* Let Q be a non-degenerate quadratic form on

V (i.e. $\mathrm{d}Q : V \to V^*$ is a bijection). If $\mu \in |V^*|$ then μe^{iQ} is integrable and

$$\int_V \mu e^{iQ} = \frac{\mu}{\mu_Q} e^{i\pi \operatorname{sgn} Q/4},$$

where sgn Q is the signature of Q and μ_Q is the unique positive element of $|V^*|$ such that

$$(\mathrm{d}Q)\,\mu_Q = \mu_Q^*.$$

In order to prove this it suffices to show that

$$(\mu e^{iQ})\hat{}\,(w) = \frac{\mu}{\mu_Q} e^{i\pi \operatorname{sgn} Q/4 - i\langle(\mathrm{d}Q)^{-1}w, w\rangle/2}.$$

The use of a canonical decomposition of V for Q:

$$V = \bigoplus_{k=1}^{n} V_k \qquad \mu e^{iQ} = \bigotimes_{k=1}^{n} \mu_k e^{iQ_k}$$

reduces the problem to the one-dimensional case, where it becomes a standard exercise in the theory of distributions.

Our next definition concerns the integration with respect to a part of the variables. In this case the objects we are going to integrate on a vector space V are elements of the space

$$S'(V)_1 \hat{\otimes} S'(E) = \mu S'(V) \hat{\otimes} S'(E) = (\mu \otimes 1)\, S'(V \oplus E)$$

(here E is a vector space and $\mu \in |V^*|^{1/2}$, $\mu \neq 0$).

DEFINITION 3. An element $\mathscr{X}$ of $S'(V)_1 \hat{\otimes} S'(E)$ is said to be *integrable on* V if

1° $\mathscr{X}_V^E \phi$ is integrable in the sense of Definition 2 for each $\phi \in S(E)$

and

2° there exists a polynomial function P on V^* and a continuous seminorm $\|\cdot\|$ on $S(E)$ such that

$$|(\mathscr{X}_V^E \phi)\hat{}\,| \leqslant P \|\phi\|$$

for any $\phi \in S(E)$. If $\mathscr{X}$ is integrable then the *integral* $\int_V \mathscr{X}$ is defined by

$$\left\langle \phi, \int_V \mathscr{X} \right\rangle = \int_V \mathscr{X}_V^E \phi = (\mathscr{X}_V^E \phi)\hat{}\,(0).$$

Remarks. (i) From 1° it follows that $\phi \longmapsto (\mathscr{T}_V^E \phi)\hat{}(w)$ is a continuous functional on $S(E)$ for each $w \in V^*$ and also that $w \longmapsto (\mathscr{T}_V^E \phi)\hat{}(w)$ is a polynomially bounded function for each $\phi \in S(E)$. Condition 2° states that the continuity in ϕ and the polynomial boundedness in w are uniform.

(ii) The space $S'(E)$ can be replaced by $S'(E)_1$, $S'(E)_0$ and similar spaces.

EXAMPLE 6. *Generalized half-densities as kernels of integral operators.* Let $\mathscr{K} \in S'(V \oplus E)$ and $\psi \in S(V)$. We shall show that $(\psi \otimes 1)\mathscr{K}$ is integrable on V and

$$\mathscr{K}_E^V \psi = \int_V (\psi \otimes 1)\mathscr{K}.$$

First observe that for any $\phi \in S(E)$

$$[(\psi \otimes 1) \cdot \mathscr{K}]_V^E \phi = \psi \cdot \mathscr{K}_V^E \phi.$$

Consequently, this generalized density is integrable (see Example 4). Using (10) we obtain

$$\{[(\psi \otimes 1) \cdot \mathscr{K}]_V^E \phi\}\hat{}(w) = \langle \overline{\chi}_w \psi, \mathscr{K}_V^E \phi\rangle = \langle \overline{\chi}_w \psi \otimes \phi, \mathscr{K}\rangle.$$

By the continuity of $\mathscr{K}$, there exist continuous seminorms $\|\cdot\|_0$ and $\|\cdot\|$ on $S(V)$ and $S(E)$ respectively, such that

$$|\langle \overline{\chi}_w \psi \otimes \phi, \mathscr{K}\rangle| \leqslant \|\overline{\chi}_w \psi\|_0 \cdot \|\phi\|.$$

Here $\|\overline{\chi}_w \psi\|_0$ is polynomially bounded as in (11).

PROPOSITION 6. (*Fubini theorem*). *Let $\mathscr{T} \in S'(V \oplus E)_1$ be integrable on V. If $V = X \oplus Y$, then $\mathscr{T}$ is integrable on X, $\int_X \mathscr{T}$ is integrable on Y and*

$$\int_Y \int_X \mathscr{T} = \int_V \mathscr{T}.$$

Proof. For $g \in S(E)_0$ we have

$$(\mathscr{T}_V^E g)\hat{} = \mathscr{T}_{V*}^{E*} \check{g}$$

since

$$\langle \zeta, (\mathscr{T}_V^E g)\hat{}\rangle = \langle \hat{\zeta}, \mathscr{T}_V^E g\rangle = \langle \hat{\zeta} \otimes g, \mathscr{T}\rangle =$$
$$= \langle (\zeta \otimes \check{g})\hat{}, \mathscr{T}\rangle = \langle \zeta \otimes \check{g}, \hat{\mathscr{T}}\rangle = \langle \zeta, \hat{\mathscr{T}}_{V*}^{E*} \check{g}\rangle$$

for $\zeta \in S(V^*)_1$. Similarly we have

$$(\mathscr{Y}_X^{Y \oplus E} f)\hat{} = \mathscr{Y}_{X^*}^{Y^* \oplus E^*} \check{\hat{f}}$$

for $f \in S(Y \oplus E)_0$. For $f = h \otimes g$, $h \in S(Y)_0$, $g \in S(E)_0$, we obtain

$$[\mathscr{Y}_X^{Y \oplus E}(h \otimes g)]\hat{} = \hat{\mathscr{Y}}_{X^*}^{Y^* \oplus E^*}(\check{\hat{h}} \otimes \check{\hat{g}}) = (\hat{\mathscr{Y}}_{V^*}^{E^*} g)_{X^*}^{Y^*} \check{\hat{h}} = [(\mathscr{Y}_V^E g)\hat{}\,]_{X^*}^{Y^*} \check{\hat{h}}.$$

Since $(\mathscr{Y}_V^E g)\hat{}$ is continuous and polynomially bounded,

$$[\mathscr{Y}_X^{Y \oplus E}(h \otimes g)]\hat{} = \int_{Y^*} (\mathscr{Y}_V^E g)\hat{}\,\check{\hat{h}} \tag{12}$$

is also a continuous and polynomially bounded function. Moreover, since there exists a polynomial P such that

$$|(\mathscr{Y}_V^E g)\hat{}\,| \leqslant P \cdot \|g\|, \quad \|\cdot\| - \text{a seminorm on } S(E)_0, \tag{13}$$

we have

$$|[\mathscr{Y}_X^{Y \oplus E}(h \otimes g)]\hat{}(p)| \leqslant \int_{Y^*} P(p, \cdot)\, \|g\| \cdot |\check{\hat{h}}(\cdot)| \leqslant P_1(p)\, \|h\|_2 \cdot \|g\| \tag{14}$$

for $p \in X^*$, where P_1 is a polynomial and $\|\cdot\|_2$ is a seminorm on $S(Y)_0$. Note that we can always choose P as a tensor product of polynomials, since there exist a positive real number C and a positive integer m such that

$$P(p, r) \leqslant C(1 + p^2 + r^2)^m \leqslant C(1 + p^2)^m (1 + r^2)^m ,$$

where $p \in X^*$, $r \in Y^*$ and the squares p^2, r^2 are taken with respect to arbitrary scalar products in X^* and Y^*.

For $f_0 \in S(Y)_0 \otimes S(E)_0$ (the algebraic tensor product) we obtain from (14)

$$|(\mathscr{Y}_X^{Y \oplus E} f_0)\hat{}\,| \leqslant P_1 \|f_0\|_1 , \tag{15}$$

where $\|\cdot\|_1$ is a continuous seminorm on $S(Y \oplus E)_0$. If we take now a sequence $f_0^k \in S(Y)_0 \otimes S(E)_0$ converging to $f \in S(Y \oplus E)_0$, then by (15) the functions $(\mathscr{Y}_X^{Y \oplus E} f_0^k)\hat{}$ converge uniformly on compact sets in X^*. Therefore the limit is a continuous function and $(\mathscr{Y}_X^{Y \oplus E} f)\hat{}$ must coincide with this function. Moreover,

$$|(\mathscr{Y}_X^{Y \oplus E} f)\hat{}\,| \leqslant P_1 \cdot \|f\|_1$$

for $f \in S(Y \oplus E)_0$, so $\mathscr{Y}$ is integrable on X. The integral $\int_X \mathscr{Y}$ is an element of $S'(Y \oplus E)_1$ given by

$$\left\langle f, \int_X \mathscr{X} \right\rangle = (\mathscr{X}_X^{Y \oplus E} f)\hat{}(0) \quad \text{for} \quad f \in S(Y \oplus E)_0.$$

Now we chek the integrability of $\int_X \mathscr{X}$ on Y. To this end we compute $[(\int_X \mathscr{X})_Y^E g]\hat{}$ for $g \in S(E)_0$:

$$\left\langle \zeta, \left[\left(\int_X \mathscr{X}\right)_Y^E g\right]\hat{} \right\rangle = \left\langle \hat{\zeta}, \left(\int_X \mathscr{X}\right)_Y^E g \right\rangle = \left\langle \hat{\zeta} \otimes g, \int_X \mathscr{X} \right\rangle =$$

$$= [\mathscr{X}_X^{Y \oplus E}(\hat{\zeta} \otimes g)]\hat{}(0) = \int_{Y^*} (\mathscr{X}_V^E g)\hat{}(0, \cdot)\, \zeta(\cdot)$$

for $\zeta \in S(Y^*)_1$ (the last equality follows from (12)). The result is a continuous, polynomially bounded function on Y^*:

$$\left[\left(\int_X \mathscr{X}\right)_Y^E g\right]\hat{}(r) = (\mathscr{X}_V^E g)\hat{}(0, r).$$

From (13) we get also the required inequality

$$\left|\left[\left(\int_X \mathscr{X}\right)_Y^E g\right]\hat{}(r)\right| \leqslant P(0, r) \cdot \|g\|$$

for $r \in Y^*$. Thus $\int_X \mathscr{X}$ is integrable on Y and we have for $g \in S(E)_0$

$$\left\langle g, \int_Y \int_X \mathscr{X} \right\rangle = \left[\left(\int_X \mathscr{X}\right)_Y^E g\right]\hat{}(0) = (\mathscr{X}_V^E g)\hat{}(0, 0) = \left\langle g, \int_V \mathscr{X} \right\rangle. \qquad \blacksquare$$

Remarks. (i) If we put $E = \{0\}$, we obtain the Fubini theorem for integrable densities on V.

(ii) The space $S'(V \oplus E)_1$ in the proposition can be easily replaced by $S'(V)_1 \hat{\otimes} S'(E)$ etc. The Proposition is formulated in terms of densities in order to simplify the notation in the proof.

4. FRESNEL DISTRIBUTIONS

a) *Regular Fresnel distributions*

DEFINITION 4. A *regular Fresnel distribution* on a vector space V is a half-density on V of the following form

$$\mathscr{K} = \mu e^{iQ}, \tag{16}$$

where Q is a quadratic function on V and $\mu \in |V^*|^{1/2}$, $\mu \neq 0$.

To each regular Fresnel distribution $\mathscr{K}$ on V there correspond a quadratic function Q on V and a regular Lagrangian subspace L of $Ph\,V$ such that (16) holds for some μ and L is generated by Q.

b) *Fresnel distributions defined by Morese families*

By Proposition 5 any (not necessarily regular) Lagrangian subspace $L \subset Ph\,V$ is generated by a Morse family M indexed over V. M is a quadratic function on $V \otimes \Lambda$, where Λ is a vector space. Therefore M corresponds to a regular Lagrangian subspace of $Ph(V \oplus \Lambda)$. Thus, one can define general Lagrangian subspaces starting from regular ones. The following proposition enables us to proceed similarly with Fresnel distributions.

PROPOSITION 7. *Let M be a quadratic function on $V \oplus \Lambda$ and let $\mu \in |V^*|^{1/2} \otimes |\Lambda^*|$, $\mu \neq 0$. Then μe^{iM} is integrable on Λ if and only if M is a Morse family of functions on Λ.* □

Proof. If M is not a Morse family then $\mathscr{X}_\phi = (\mu e^{iM})^V_\Lambda\, \phi$ is invariant with respect to a one-parameter group of translations

$$\Lambda \ni \lambda \longmapsto \lambda + t\lambda_0 \in \Lambda \quad (\lambda_0 \in \Lambda, \lambda_0 \neq 0, t \in \mathbb{R})$$

for $\phi \in S(V)$ (see condition (i) of Proposition 3). Therefore the Fourier transform of $\mathscr{X}_\phi$ satisfies the equation

$$\langle \lambda_0, \cdot \rangle\, \hat{\mathscr{X}}_\phi = 0$$

and $\hat{\mathscr{X}}_\phi$ cannot be a continuous function, unless it is zero. But it is not zero for a certain $\phi \in S(V)$, since $\mu e^{IM} \neq 0$. Thus μe^{iM} is not integrable.

Now, let us assume that M is a Morse family on V. Let M be given by formula (2) with A and B satisfying the condition (iii) of Proposition 3:

$$\operatorname{Im} A + \operatorname{Im} B = \Lambda^*. \tag{17}$$

Let $\mu = \mu_V \otimes \mu_\Lambda$, where $\mu_V \in |V^*|^{1/2}$, $\mu_\Lambda \in |\Lambda^*|$ and let $a(\lambda) = 1/2\,\langle \lambda, A\lambda \rangle$.

We have for $\phi \in S(V)$

$$\mathscr{X}_{\phi}(\lambda) = (\mu e^{iM})^{V}_{\Lambda}\,\phi(\lambda) = \mu_{\Lambda}\, e^{ia(\lambda)} \int_{V/\ker B} e^{-i\langle \cdot ,\, B^{*}\lambda\rangle}\phi_B ,$$

where

$$\phi_B = \int_{\ker B} \mu_V e^{iQ_0}\phi$$

is an element of $S(V/\ker B)_1$. Thus

$$\mathscr{X}_{\phi}(\lambda) = \mu_{\Lambda}\, e^{ia(\lambda)}\hat{\phi}_B(B^{*}\lambda) = \mu_{\Lambda}\, e^{ia(\lambda)}(\phi_B \circ B_0^{-1})\hat{}\,(\lambda),$$

where $B_0 : V/\ker B \to \operatorname{Im} B$ is the isomorphism induced by B. Using the result of Example 3 we can write

$$(\phi_B \circ B_0^{-1})\hat{}\,(\lambda) = \hat{\phi}_{\Lambda^{*}}(\lambda),$$

where $\phi_{\Lambda^{*}} = (\phi_B \circ B_0^{-1}) \cdot \delta_{\operatorname{Im} B}$, and also

$$e^{ia(\lambda)} = (\mathscr{X}\delta_{\operatorname{Im} A})\hat{}\,(\lambda),$$

where $\mathscr{X} \in S'(\operatorname{Im} A)_1$ is given by

$$\mathscr{X}(\varkappa) = \mu^{*}_{a_0} e^{i\pi \operatorname{sgn}(a)/4 - i\langle A_0^{-1}\varkappa,\, \varkappa\rangle/2}$$

for $\varkappa \in \operatorname{Im} A$ (see Example 5). Here $A_0 : \Lambda/\ker A \to \operatorname{Im} A$ is the isomorphism induced by A and $a_0(\cdot) = \dfrac{1}{2}\langle \cdot, A_0 \cdot\rangle$.

Now we calculate the Fourier transform of $\mathscr{X}_{\phi}$. For $\zeta \in S(\Lambda^{*})_1$,

$$\begin{aligned}
(18)\qquad \langle \hat{\zeta}, \mathscr{X}_{\phi}\rangle &= \langle \hat{\zeta}, \hat{\phi}_{\Lambda^{*}}\mu_{\Lambda} e^{ia}\rangle = \langle \hat{\zeta}\hat{\phi}_{\Lambda^{*}}, \mu_{\Lambda} e^{ia}\rangle = \\
&= \langle (\zeta * \phi_{\Lambda^{*}})\hat{}\,, \mu_{\Lambda} e^{ia}\rangle = \langle \zeta * \phi_{\Lambda^{*}}, (\mu_{\Lambda} e^{ia})\hat{}\,\rangle = \\
&= \langle \zeta * \phi_{\Lambda^{*}}, (e^{ia})\hat{}/\mu^{*}_{\Lambda}\rangle = \langle (\zeta * \phi_{\Lambda^{*}})/\mu^{*}_{\Lambda}, \mathscr{X}\delta_{\operatorname{Im} A}\rangle = \\
&= \langle [(\zeta * \phi_{\Lambda^{*}})/\mu^{*}_{\Lambda}]|_{\operatorname{Im} A}, \mathscr{X}\rangle = \\
&= \int_{\operatorname{Im} A} \mathscr{X}(\varkappa_A) \int_{\operatorname{Im} B} \phi_B(B_0^{-1}\varkappa_B)\zeta(\varkappa_A - \varkappa_B)/\mu^{*}_{\Lambda} = \\
&= \int_{\operatorname{Im} A \times \operatorname{Im} B} \mathscr{X}(\varkappa_A)\phi_B(B_0^{-1}\varkappa_B)\zeta(\varkappa_A - \varkappa_B)/\mu^{*}_{\Lambda} \\
&\qquad\qquad (\varkappa_A \in \operatorname{Im} A,\ \varkappa_B \in \operatorname{Im} B).
\end{aligned}$$

We have used the standard theorem on the Fourier transform of a convolution and the result of Example 2.

Now consider the exact sequence of linear mappings

$$0 \longrightarrow \operatorname{Im} A \cap \operatorname{Im} B \xrightarrow{\alpha} \operatorname{Im} A \times \operatorname{Im} B \xrightarrow{\beta} \operatorname{Im} A + \operatorname{Im} B \longrightarrow 0,$$

where

$$\alpha(\eta) = (\eta, \eta) \quad \text{for} \quad \eta \in \operatorname{Im} A \cap \operatorname{Im} B$$

$$\beta(\varkappa_A, \varkappa_B) = \varkappa_A - \varkappa_B \quad \text{for} \quad \varkappa_A \in \operatorname{Im} A, \varkappa_B \in \operatorname{Im} B.$$

Condition (17) implies

$$(\operatorname{Im} A \times \operatorname{Im} B)/(\operatorname{Im} A \cap \operatorname{Im} B) = \Lambda^*.$$

It follows that the last integral in (18) can be written in the form $\int_{\Lambda^*} \zeta(\varkappa) f_\phi(\varkappa)$, where

(19) $$f_\phi(\varkappa) = \int_{\operatorname{Im} A \cap \operatorname{Im} B} \mu_0 e^{i\pi \operatorname{sgn}(a)/4 - i\langle A_0^{-1}(\varkappa_A + \eta), \varkappa_A + \eta\rangle/2} \phi_B(B_0^{-1}(\varkappa_B + \eta)).$$

Here $\varkappa_A \in \operatorname{Im} A$, $\varkappa_B \in \operatorname{Im} B$ are such that $\varkappa_A - \varkappa_B = \varkappa$, μ_1 is an element of $|(V/\ker B)^*|$ different from zero and $\mu_0 \in |(\operatorname{Im} A \cap \operatorname{Im} B)^*|$ is defined by $\mu_{a_0}^* \otimes B_0 \mu_1 = \mu_0 \otimes \mu_\Lambda^*$.

It is easy to see that $\hat{\mathscr{Y}}_\phi$ is just the function f_ϕ. The integral on the right hand side of (19) is a continuous function of $\varkappa_A$ and $\varkappa_B$. It follows that $\hat{\mathscr{Y}}_\phi(\varkappa)$ is a continuous function of $\varkappa$. We have also

$$|\hat{\mathscr{Y}}_\phi(\varkappa)| \leqslant \|\phi\|,$$

where $\|\phi\| = \sup_{\varkappa_B \in \operatorname{Im} B} \int_X \mu_0 |\phi_B(B_0^{-1}(\varkappa_B + \eta))|$ $(X = \operatorname{Im} A \cap \operatorname{Im} B)$ is a continuous seminorm on $S(V)$. ■

DEFINITION 5. An element $\mathscr{K}$ of $S'(V)$ is called a *Fresnel distribution* on V if $\mathscr{K} = \int_\Lambda \mu e^{iM}$, where μ and M satisfy the assumptions of Proposition 7.

Examples of Fresnel distributions are provided by regular Fresnel distributions (take $\Lambda = \{0\}$).

c) *Fresnel distributions characterized in terms of constraints*

PROPOSITION 8. *Each Fresnel distribution $\mathscr{K}$ on a vector space V is of the following form*

$$(20)\qquad \langle \psi, \mathscr{K} \rangle = \int_C \nu \circ (\psi|_C) e^{iQ},$$

where $\psi \in S(V)$

C is a subspace of V

Q is a quadratic function on C

$\nu \in \mathrm{Hom}\,(|V^*|^{1/2}, |C^*|), \nu \neq 0$

and $\psi|_C$ *denotes the restriction of* ψ *to* C.

The objects ν, C and Q are uniquely determined by $\mathscr{K}$. If $\mathscr{K} = \int_\Lambda \mu e^{iM}$ then C and Q are determined by M as in Proposition 4.

Proof. From formula (19) we have for $\phi \in S(V)$

$$\left\langle \phi, \int_\Lambda \mu e^{iM} \right\rangle = \mathscr{T}_\phi(0) =$$

$$= \int_{\mathrm{Im}\,A \cap \mathrm{Im}\,B} \mu_0 e^{i\pi\,\mathrm{sgn}\,(a)/4 - \frac{i}{2}\langle A_0^{-1}\eta, \eta\rangle} \phi_B(B_0^{-1}\eta)/\mu_1 =$$

$$= \int_{B_0^{-1}(\mathrm{Im}\,A)} (B_0^{-1}\mu_0) e^{i\pi\,\mathrm{sgn}\,(a)/4 - \frac{i}{2}\langle A_0^{-1}B_0\,\cdot\,, B_0\,\cdot\,\rangle} \phi_B/\mu_1 =$$

$$= \int_{B_0^{-1}(\mathrm{Im}\,A)} (B_0^{-1}\mu_0) e^{i\pi\,\mathrm{sgn}\,(a)/4 - \frac{i}{2}\langle A_0^{-1}B_0\,\cdot\,, B_0\,\cdot\,\rangle} \left(\int_{\ker B} \mu_V e^{iQ_0} \right) / \mu_1 =$$

$$= \int_C \mu_C e^{i\pi\,\mathrm{sgn}\,(a)/4 - \langle A_0^{-1}Bv, Bv\rangle/2 + iQ_0(v)} \phi(v)/\mu_V,$$

where $C = B^{-1}(\mathrm{Im}\,A)$ and $\mu_C = \mu_2 \otimes B_0^{-1}\mu_0 \in |C^*|$. Here $\mu_2 \in |(\ker B)^*|$ is such that $\mu_2 \otimes \mu_1 = \mu_V^2$. It follows that $\mathscr{K}$ has the form (20) with C and Q determined by M as in Proposition 4. Uniqueness of ν, C and Q is obvious. ∎

5. FRESNEL KERNELS

DEFINITION 6. Let X and Y be two vector spaces. A *Fresnel kernel from X to Y* is a Fresnel distribution on $Y \oplus X$.

The set of Fresnel kernels from X to Y will be denoted by $FK(Y, X)$.

To each Fresnel kernel $\mathscr{F}$ there corresponds a morphism Q (or ρ) in the category QF (or LPR). We shall write

$$Q = cl(\mathscr{F}) \text{ (or } \rho = cl(\mathscr{F}))$$

in order to denote the morphism Q (or ρ) corresponding to $\mathscr{F}$.

By example 6 a Fresnel kernel $\mathscr{F} \in FK(Y, X)$ is the integral kernel of the operator $\mathscr{F}^X_Y : S(X) \to S'(Y)$. The adjoint (in the sense of the scalar product (5)) operator $(\mathscr{F}^X_Y)^\dagger : S(Y) \to S'(X)$ corresponds to the *adjoint* of $\mathscr{F}$, which is by definition a Fresnel kernel $\mathscr{F}^\dagger$ from Y to X such that

$$\langle \psi \otimes \phi, \mathscr{F}^\dagger \rangle = \overline{\langle \overline{\phi} \otimes \overline{\psi}, \mathscr{F} \rangle} = \langle \phi \otimes \psi, \overline{\mathscr{F}} \rangle$$

for $\psi \in S(X)$, $\phi \in S(Y)$. We have then

$$(\mathscr{F}^\dagger)^Y_X = (\mathscr{F}^X_Y)^\dagger$$

and

$$cl(\mathscr{F}^\dagger) = cl(\mathscr{F})^\dagger .$$

We shall use occasionally the following abbreviated notation

$$\mathscr{F}_+ = \mathscr{F}^X_Y \quad \text{and} \quad \mathscr{F}_- = \mathscr{F}^Y_X \tag{21}$$

for a Fresnel kernel $\mathscr{F} \in FK(Y, X)$.

a) *Composition*

PROPOSITION 9. *Let $\rho_1 : PhX \to PhV$ and $\rho_2 : PhV \to PhY$ be two linear phase relations. Let $M_1 : V \oplus X \oplus \Lambda_1 \to \mathbb{R}$ and $M_2 : Y \oplus V \oplus \Lambda_2 \to \mathbb{R}$ be Morse families generating ρ_1 and ρ_2, respectively. Then the function $M : Y \oplus X \oplus \Lambda_1 \oplus \Lambda_2 \oplus V \to \mathbb{R}$, given by*

$$M(y, x, \lambda_1, \lambda_2, v) = M_2(y, v, \lambda_2) + M_1(v, x, \lambda_1),$$

is a Morse family indexed over $Y \oplus X$ if and only if

$$\rho_1(\{0\}) \cap \rho_2^\dagger(\{0\}) = \{0\}. \tag{22}$$

In this case M generates $\rho_2 \circ \rho_1$.

Proof. We have

$$\rho_1(\{0\}) = \{v \oplus w \in V \oplus V^* \mid \text{there exists } \lambda_1 \in \Lambda_1 \text{ such that } \mathrm{d}M_1(v, 0, \lambda_1) = (w, 0, 0)\}$$

and

$$\rho_2^\dagger(\{0\}) = \{v \oplus w \in V \oplus V^* \mid \text{there exists } \lambda_2 \in \Lambda_2 \text{ such that } \mathrm{d}M_2(0, v, \lambda_2) = (0, -w, 0)\}.$$

Hence, condition (22) is equivalent to the following condition:

(23) $\mathrm{d}M_1(v, 0, \lambda_1) = (w, 0, 0)$ and $\mathrm{d}M_2(0, v, \lambda_2) = (0, -w, 0)$ implies $v = 0$, $w = 0$ for $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$, $v \in V$, $w \in V^*$.

We have also

$$\langle(\tilde{y}, \tilde{x}, \tilde{\lambda}_1, \tilde{\lambda}_2, v), \mathrm{d}M(y, x, \lambda_1, \lambda_2, v)\rangle = \\ = \langle(\tilde{y}, \tilde{v}, \tilde{\lambda}_2), \mathrm{d}M_2(y, v, \lambda_2)\rangle + \langle(\tilde{v}, \tilde{x}, \tilde{\lambda}_1), \mathrm{d}M_1(v, x, \lambda_1)\rangle.$$

We first prove the sufficiency of condition (22). Suppose that $\mathrm{d}M(0, 0, \lambda_1, \lambda_2, v) = 0$. Then

$$\langle(\tilde{y}, \tilde{v}, \tilde{\lambda}_2), \mathrm{d}M_2(0, v, \lambda_2)\rangle + \langle(\tilde{v}, \tilde{x}, \tilde{\lambda}_1), \mathrm{d}M_1(v, 0, \lambda_1)\rangle = 0$$

for each $\tilde{y}, \tilde{x}, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{v}$. If we set $\tilde{y} = 0, \tilde{v} = 0$ and $\tilde{\lambda}_2 = 0$, we obtain $\mathrm{d}M_1(v, 0, \lambda_1) = (w_1, 0, 0)$ for some $w_1 \in V^*$. If we set $\tilde{v} = 0$, $\tilde{x} = 0$ and $\tilde{\lambda}_1 = 0$, we obtain $\mathrm{d}M_2(0, v, \lambda_2) = (0, w_2, 0)$ for some $w_2 \in V^*$. We see that $w_1 + w_2 = 0$. From (23) we obtain $v = 0$, $w_1 = 0$ and $w_2 = 0$, hence $\mathrm{d}M_1(0, 0, \lambda_1) = 0$ and $\mathrm{d}M_2(0, 0, \lambda_2) = 0$. Since M_1 and M_2 are Morse families it follows that $\lambda_1 = 0$ and $\lambda_2 = 0$.

Now we prove the necessity of condition (22). Suppose that $\mathrm{d}M_1(v, 0, \lambda_1) = (w, 0, 0)$ and $\mathrm{d}M_2(0, v, \lambda_2) = (0, -w, 0)$. This gives

$$\langle(\tilde{y}, \tilde{x}, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{v}), \mathrm{d}M(0, 0, \lambda_1, \lambda_2, v)\rangle = 0$$

for each $\tilde{y}$, $\tilde{x}$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{v}$, hence $\lambda_1 = 0$, $\lambda_2 = 0$ and $v = 0$ because M is assumed to be a Morse family indexed over $Y \oplus X$. Consequently $w = 0$.

The last assertion of the proposition may be proved as follows. Let $\rho : Ph\, V \to Ph\, Y$ be the relation generated by M. Then $(y, r, x, p) \in \mathrm{Graph}\, \rho$ if and only if there exists λ_1, λ_2 and v such that $\mathrm{d}M(y, x, \lambda_1, \lambda_2, v) = (-p, r, 0, 0, 0,)$, or if and only if there exist λ_1, λ_2 and v such that

(24) $\langle(\tilde{y}, \tilde{v}, \tilde{\lambda}_2), \mathrm{d}M_2(y, v, \lambda_2)\rangle + \langle(\tilde{v}, \tilde{x}, \tilde{\lambda}_1), \mathrm{d}M_1(v, x, \lambda_1)\rangle = \langle\tilde{y}, r\rangle - \langle\tilde{x}, p\rangle$

for each $\tilde{y}$, $\tilde{x}$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{v}$. On the other hand, $(y, r, x, p) \in \text{Graph}\,(\rho_2 \circ \rho_1)$ if and only if there exist λ_1, λ_2, v and w such that

(25) $$\mathrm{d}M_1(v, x, \lambda_1) = (0, -p, w) \ \text{ and } \ \mathrm{d}M_2(y, v, \lambda_2) = (r, -w, 0).$$

It is easy to see that conditions (24) and (25) are equivalent. ∎

DEFINITION 7. Two Fresnel kernels $\mathscr{F}_1 \in FK(V, X)$ and $\mathscr{F}_2 \in FK(Y, V)$ are said to be *composable* if for each representations

(26) $$\mathscr{F}_1 = \int_{\Lambda_1} \mu_1 e^{iM_1} \qquad \mathscr{F}_2 = \int_{\Lambda_2} \mu_2 e^{iM_2},$$

where M_1 and M_2 are Morse families indexed over $V \oplus X$ and $Y \oplus V$, resp., and $\mu_1 \in |(V \oplus X)^*|^{1/2} \otimes |\Lambda_1^*|$, $\mu_2 \in |(Y \oplus V)^*|^{1/2} \otimes |\Lambda_2^*|$, the density

$$(y, x, \lambda_1, \lambda_2, v) \longmapsto \mu_2 \mu_1 e^{iM_2(y, v, \lambda_2)} e^{iM_1(v, x, \lambda_1)}$$

is integrable on $\Lambda_1 \oplus \Lambda_2 \oplus V$. Here $\mu_2 \mu_1$ is considered as an element of $|(Y \oplus X)^*|^{1/2} \otimes |(\Lambda_1 \oplus \Lambda_2 \oplus V)^*|$.

By Proposition 9, two Fresnel kernels $\mathscr{F}_1 \in FK(V, X)$ and $\mathscr{F}_2 \in FK(Y, V)$ are composable if and only if the relations $\rho_1 = cl(\mathscr{F}_1)$ and $\rho_2 = cl(\mathscr{F}_2)$ satisfy condition (22). If $\mathscr{F}_1$ and $\mathscr{F}_2$ are composable then the integral

(27) $$\int_{\Lambda_1 \oplus \Lambda_2 \oplus V} \mu_2 \mu_1 e^{iM_2(y, v, \lambda_2)} e^{iM_1(v, x, \lambda_1)}$$

does not depend on the choice of representations (26) of $\mathscr{F}_1$ and $\mathscr{F}_2$ by Morse families. This follows from the Fubini theorem (Proposition 6). Indeed,

$$\int_{\Lambda_1 \oplus \Lambda_2 \oplus V} \mu_2 \mu_1 e^{iM_2(y, v, \lambda_2)} e^{iM_1(v, x, \lambda_1)} =$$

$$= \int_{\Lambda_2 \oplus V} \mu_2 e^{iM_2(y, v, \lambda_2)} \mathscr{F}_1(v, x)$$

does not depend on the choice of the representation of $\mathscr{F}_1$; similar reasoning applies to $\mathscr{F}_2$.

DEFINITION 8. The *composition* $\mathscr{F}_2 \circ \mathscr{F}_1$ of two composable Fresnel Kernels $\mathscr{F}_1 \in FK(V, X)$ and $\mathscr{F}_2 \in FK(Y, V)$ is the integral (27) constructed for arbitrary representations (26) of $\mathscr{F}_1$ and $\mathscr{F}_2$.

It is immediately seen from Proposition 9 that

$$cl(\mathscr{F}_2 \circ \mathscr{F}_1) = cl(\mathscr{F}_2) \circ cl(\mathscr{F}_1) \tag{28}$$

for two composable kernels $\mathscr{F}_1$ and $\mathscr{F}_2$.

If $cl(\mathscr{F}_1)$ is an epimorphism the $\mathscr{F}_2 \circ \mathscr{F}_1$ is always defined since condition (22) is always fulfilled in this case. We shall show that in this case $\mathscr{F}_2 \circ \mathscr{F}_1$ corresponds to the composition of operators $(\mathscr{F}_1)^X_V$ and $(\mathscr{F}_2)^V_Y$. To this end we need the following lemmas.

LEMMA 1. *Let* $\mathscr{F} \in FK(Y, X)$. *If* $cl(\mathscr{F})$ *is an epimorphism then* $\mathscr{F}^X_Y(S(X)) = S(Y)$.

Proof. Let $Q = cl(\mathscr{F})$ be defined on $C \subset Y \oplus X$. Set

$$C_X = \text{the projection of } C \text{ in } X,$$

$$C_Y = \text{the projection of } C \text{ in } Y,$$

$$X_0 = \{x \in X \mid 0 \oplus x \in C\},$$

$$Y_0 = \{y \in Y \mid y \oplus 0 \in C\}.$$

Since Q is an epimorphism, $C_Y = Y$. C induces a linear isomorphism of C_X/X_0 and $C_Y/Y_0 = Y/Y_0$. This isomorphism induces a linear isomorphism $A : Y_1 \to X_1$, where $X_1 \subset C_X$ and $Y_1 \subset Y$ are such that

$$C_X = X_0 \oplus X_1, \quad Y = Y_0 \oplus Y_1.$$

We can therefore parametrize C by $Y_0 \oplus Y_1 \oplus X_0$:

$$C = \{y_0 \oplus y_1 \oplus x_0 \oplus Ay_1 \in Y \oplus C_X \mid y_0 \in Y_0, y_1 \in Y_1, x_0 \in X_0\}.$$

In this parametrization Q has the following form

$$Q(y_0, y_1, x_0) = a(x_0) + \langle x_0, By_0 \rangle +$$

$$+ c(y_0) + q(y_1) + \langle x_0, Ey_1 \rangle + \langle y_0, Fy_1 \rangle,$$

where a, c, q are quadratic functions and $B : Y_0 \to X_0^*$, $E : Y_1 \to X_0^*$, $F : Y_1 \to Y_0^*$. The variation with respect to $\tilde{y}_0 \in Y_0$ gives

$$Y_0^* \ni r_0 = B^* x_0 + \mathrm{d}c(y_0) + Fy_1$$

and this is a part of equations for the graph of the relation $\rho = cl(\mathscr{F})$. This equation has to be satisfied for each $r_0 \in Y_0^*$, $y_0 \in Y_0$ and $y_1 \in Y_1$ by a suitable $x_0 \in X_0$ since ρ is an epimorphism. This implies that B^* is surjective and B is injective.

Let us choose positive elements $\mu_X \in |X^*|^{1/2}$ and $\mu_Y \in |Y^*|^{1/2}$. Let $\psi = f\mu_X \in S(X)$ and $\phi \in S(Y)$. Then

$$\langle \phi, \mathscr{F}_Y^X \psi \rangle = \langle \psi \otimes \phi, \mathscr{F} \rangle = \int_C \nu \circ ((\phi \otimes \psi)|_C) e^{iQ} =$$

$$= \int_{Y_0 \oplus Y_1 \oplus X_0} \mu \, \mu_Y \phi(y_0, y_1) f(x_0, Ay_1) \, e^{iQ(y_0, y_1, x_0)}$$

for certain $\nu \in \mathrm{Hom}\,(|(Y \oplus X)^*|^{1/2}, |C^*|)$, $\mu \in |X_0^*|$. We have then

$$\langle \phi, \mathscr{F}_Y^X \psi \rangle = \int_{Y_0 \oplus Y_1} \mu_Y \phi(y_0, y_1) e^{i[c(y_0) + q(y_1) + \langle y_0, Fy_1 \rangle]} \times$$

$$\times \int_{X_0} [\mu f(x_0, Ay_1) e^{i(a(x_0) + \langle x_0, Ey_1 \rangle)}] e^{i\langle x_0, By_0 \rangle}.$$

The mapping $S(X) \ni \psi \longmapsto \xi \in \mu \cdot S(Y_1 \oplus X_0)_0$, where

$$\xi(y_1, x_0) = \mu f(x_0, Ay_1) e^{i(a(x_0) + \langle x_0, Ex_1 \rangle)},$$

is a bijection of Schwartz spaces. The integral

$$\eta(y_0, y_1) = \int_{X_0} \xi(y_1, x_0) e^{i\langle x_0, By_0 \rangle}$$

is the partial (with respect to x_0 only) Fourier transform of ξ, evaluated at $p_0 = -By_0$. By the injectivity of B, the mapping $\mu \cdot S(Y_1 \oplus X_0)_0 \ni \xi \longmapsto \eta \in S(Y_0 \oplus Y_1)_0$ is surjective. Finally, the mapping $S(Y_0 \oplus Y_1)_0 \ni \eta \longmapsto \vartheta \in S(Y)$, where

$$\vartheta = \mu_Y \eta e^{i[c + q + \langle \cdot, F \cdot \rangle]},$$

is a bijection of Schwartz spaces. We conclude that

$$S(X) \ni \psi \longmapsto \mathscr{F}_Y^X \psi = \vartheta \in S(Y)$$

is a surjective mapping. ■

It follows from Lemma 1 that if $\mathscr{F}$ satisfies the conditions stated in this lemma and $\tilde{\mathscr{F}} \in FK(V, Y)$, then $\tilde{\mathscr{F}}_+ \circ \mathscr{F}_+$ is well defined as the composition of operators.

LEMMA 2. *If* $\mathscr{X} \in S'(V \oplus \Lambda)_1$ *is integrable on* Λ *and* $f \in S(V)_0$, *then* $(f \otimes 1)\mathscr{X}$ *is an integrable density on* $V \oplus \Lambda$.

Proof. With χ defined as in Example 4 the function

$$V^* \oplus \Lambda^* \longrightarrow \mathbb{C}, \ (w, \varkappa) \longmapsto [\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa)$$

is polynomially bounded:

$$|[\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa)| \leqslant P(\varkappa) \cdot \|\overline{\chi}_w f\| < P(\varkappa) P_1(w) \|f\|_1$$

(see formula (11)). It is also continuous. This follows from the inequality

$$|[\mathscr{X}_\Lambda^V(\overline{\chi}_{w_k} f)](\varkappa_k) - [\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa)| \leqslant$$
$$\leqslant |[\mathscr{X}_\Lambda^V(\overline{\chi}_{w_k} f - \overline{\chi}_w f)]\hat{}(\varkappa)| + |[\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa_k) - [\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa)| \leqslant$$
$$\leqslant P(\varkappa_k) \|\overline{\chi}_{w_k} f - \overline{\chi}_w f\| + |[\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa_k) - [\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa)|,$$

the continuity of $V^* \ni w \longmapsto \overline{\chi}_w f \in S(V)_0$ and the continuity of $[\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa)$ with respect to $\varkappa$. We shall show that

$$[(f \otimes 1)\mathscr{X}]\hat{}(w, \varkappa) = [\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa).$$

Indeed, from formula (10) we have for any $\zeta \in S(V^*)_1$, $\eta \in S(\Lambda^*)_1$

$$\langle \zeta \otimes \eta, [(f \otimes 1)\mathscr{X}]\hat{}\rangle = \langle \hat{\zeta} \otimes \hat{\eta}, (f \otimes 1)\mathscr{X}\rangle = \langle f\hat{\zeta} \otimes \hat{\eta}, \mathscr{X}\rangle =$$
$$= \langle f\hat{\zeta}, \mathscr{X}_V^\Lambda \hat{\eta}\rangle = \langle \hat{\zeta}, f\mathscr{X}_V^\Lambda \hat{\eta}\rangle = \langle \zeta, [f\mathscr{X}_V^\Lambda \hat{\eta}]\hat{}\rangle =$$
$$= \int_{V^*} \zeta(w) \langle \overline{\chi}_w f, \mathscr{X}_V^\Lambda \hat{\eta}\rangle = \int_{V^*} \zeta(w) \langle \hat{\eta}, \mathscr{X}_V^\Lambda(\overline{\chi}_w f)\rangle =$$
$$= \int_{V^*} \zeta(w) \langle \eta, [\mathscr{X}_V^\Lambda(\overline{\chi}_w f)]\hat{}\rangle = \int_{V^* \oplus \Lambda^*} \eta(\varkappa)\zeta(w)[\mathscr{X}_\Lambda^V(\overline{\chi}_w f)]\hat{}(\varkappa).$$ ■

PROPOSITION 10. *If* $\mathscr{F}_1 \in FK(V, X)$, $\mathscr{F}_2 \in FK(Y, V)$ *and* $cl(\mathscr{F}_1)$ *is an epimorphism, then*

$$(\mathscr{F}_2 \circ \mathscr{F}_1)^X_Y = (\mathscr{F}_2)^V_Y \circ (\mathscr{F}_1)^X_V .$$

Proof. Let $\psi \in S(X)$ and $\phi \in S(Y)$. It follows from Lemma 2 that for any representation (26) of $\mathscr{F}_1$ and $\mathscr{F}_2$, the density $\mathscr{X}$ on $W = Y \oplus X \oplus \Lambda_1 \oplus \Lambda_2 \oplus V$ given by

$$\mathscr{X} = \phi(y)\, \psi(x)\, \mu_2 e^{iM_2(y,v,\lambda_2)} \mu_1 e^{iM_1(v,x,\lambda_1)}$$

is integrable. From the Fubini theorem we have

$$\int_W \mathscr{X} = \int_{Y \oplus V \oplus \Lambda_2} \phi(y) \mu_2 e^{iM_2(y,v,\lambda_2)} \int_X \int_{\Lambda_1} \mu_1 e^{iM_1(v,x,\lambda_1)} =$$

$$= \int_Y \int_V \int_{\Lambda_2} \phi(y) \mu_2 e^{iM_2(y,v,\lambda_2)} ((\mathscr{F}_1)^X_V \psi)(v) =$$

$$= \int_Y \phi(y) [(\mathscr{F}_2)^V_Y (\mathscr{F}_1)^X_V \psi](y) = \langle \phi, (\mathscr{F}_2)^V_Y (\mathscr{F}_1)^X_V \psi \rangle.$$

On the other hand,

$$\int_W \mathscr{X} = \int_{Y \oplus X} \phi(y) \psi(x) \int_{\Lambda_1 \oplus \Lambda_2 \oplus V} \mu_2 e^{iM_2(y,v,\lambda_2)} \mu_1 e^{iM_1(v,x,\lambda_1)} =$$

$$= \langle \phi \otimes \psi, \mathscr{F}_2 \circ \mathscr{F}_1 \rangle = \langle \phi, (\mathscr{F}_2 \circ \mathscr{F}_1)^X_Y \psi \rangle. \qquad \blacksquare$$

b) *The partial category of Fresnel kernels.*

We shall show that vector spaces with Fresnel kernels as morphisms satisfy all requirements to form a category, exept that one stating that the composition is defined for *each* pair $\mathscr{F}_1$, $\mathscr{F}_2$ such that $\mathscr{F}_1 \in FK(V, X)$ and $\mathscr{F}_2 \in \in FK(Y, V)$. We say that vector spaces with Fresnel kernels form a *partial category*.

First of all, there exists the *identity morphism* $I_V \in FK(V, V)$ i.e. a morphism which is composable with any morphism $\mathscr{F}_1 \in FK(V, X)$ and any morphism $\mathscr{F}_2 \in FK(Y, V)$ (in the correct order) and satisfies

$$I_V \circ \mathscr{F}_1 = \mathscr{F}_1, \quad \mathscr{F}_2 \circ I_V = \mathscr{F}_2 .$$

This morphism I_V is the Fresnel kernel which corresponds to the identity $(I_V)_+$ of $S(V)$:

$$\langle \phi \otimes \psi, I_V \rangle = \langle \phi, \psi \rangle \text{ for } \phi, \psi \in S(V).$$

It is easy to see that $\langle \psi, I_V \rangle = \int_D \nu \circ (\psi |_D)$, where D is the diagonal of $V \oplus V$ and ν is the composition of the following obvious mappings:

$$| V^* |^{1/2} \otimes | V^* |^{1/2} \longrightarrow | V^* | \longrightarrow | D^* |.$$

The *associativity* holds in the sense of the following

PROPOSITION 11. *If $\mathscr{F}_2 \circ \mathscr{F}_1$ is defined and $\mathscr{F}_3 \circ (\mathscr{F}_2 \circ \mathscr{F}_1)$ is defined then also $\mathscr{F}_3 \circ \mathscr{F}_2$, $(\mathscr{F}_3 \circ \mathscr{F}_2) \circ \mathscr{F}_1$ are defined and*

$$\mathscr{F}_3 \circ (\mathscr{F}_2 \circ \mathscr{F}_1) = (\mathscr{F}_3 \circ \mathscr{F}_2) \circ \mathscr{F}_1.$$

Proof. For $k = 1, 2, 3$ let $\mathscr{F}_k \in FK(V_{k+1}, V_k)$, $\rho_k = cl(\mathscr{F}_k)$ and let $\mathscr{F}_k = \int_{\Lambda_k} \mu_k e^{iM_k}$ be representations of $\mathscr{F}_k$ by Morse families. Then

$$\mathscr{X}_{12}(v_3, v_1, \lambda_1, \lambda_2, v_2) = \mu_2 e^{iM_2(v_3, v_2, \lambda_2)} \mu_1 e^{iM_1(v_2, v_1; \lambda_1)}$$

is integrable on $\Lambda_1 \oplus \Lambda_2 \oplus V_2$ and $\int_{\Lambda_1 \oplus \Lambda_2 \oplus V_2} \mathscr{X}_{12} = \mathscr{F}_2 \circ \mathscr{F}_1$. Also

$$\mathscr{X}(v_4, v_1, \lambda_1, \lambda_2, \lambda_3, v_3, v_2) = \\ = \mu_3 e^{iM_3(v_4, v_3, \lambda_3)} \mu_2 e^{iM_2(v_3, v_2, \lambda_2)} \mu_1 e^{iM_1(v_2, v_1, \lambda_1)}$$

is integrable on $W = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \oplus V_3 \oplus V_2$ and $\int_W \mathscr{X} = \mathscr{F}_3 \circ (\mathscr{F}_2 \circ \mathscr{F}_1)$.

Since $\rho_2(\{0\}) \subset (\rho_2 \circ \rho_1)(\{0\})$, we have $\rho_3^\dagger(\{0\}) \cap \rho_2(\{0\}) = \{0\}$. Hence $\mathscr{F}_3 \circ \mathscr{F}_2$ is defined and

$$(\mathscr{F}_3 \circ \mathscr{F}_2)(v_4, v_2) = \int_{\Lambda_2 \oplus \Lambda_3 \oplus V_3} \mu_3 e^{iM_3(v_4, v_3; \lambda_3)} \mu_2 e^{iM_2(v_3, v_2, \lambda_2)}.$$

Since $\mathscr{X}$ is integrable on W, the composition $(\mathscr{F}_3 \circ \mathscr{F}_2) \circ \mathscr{F}_1$ is defined and equal to $\int_W \mathscr{X}$. ∎

The map

$$V \longmapsto V^\dagger = V, \quad \mathscr{F} \longmapsto \mathscr{F}^\dagger,$$

defines a contravariant *functor* in our partial category in the sense that whenever $\mathscr{F}_2 \circ \mathscr{F}_1$ is defined, $\mathscr{F}_1^\dagger \circ \mathscr{F}_2^\dagger$ is also defined (since condition (22) is satisdied) and

$$(\mathscr{F}_2 \circ \mathscr{F}_1)^\dagger = \mathscr{F}_1^\dagger \circ \mathscr{F}_2^\dagger .$$

This equality follows from the fact if $\mathscr{F}(y,x) = \int_\Lambda \mu e^{iM(y,x,\lambda)}$ is a representation of $\mathscr{F}$ by a Morse family, then

$$\mathscr{F}^\dagger(x,y) = \int_\Lambda \bar{\mu} e^{-iM(y,x,\lambda)}$$

is a representation of $\mathscr{F}^\dagger$ by a Morse family.

The map

$$V \longmapsto Ph V, \qquad \mathscr{F} \longmapsto cl(\mathscr{F}),$$

defines a covariant functor from the partial category to the category *LSR* (formula (28)).

c) *Special morphisms*

PROPOSITION 12. *For each Fresnel kernel* $\mathscr{F} \in FK(Y,X)$ *the following conditions are equivalent:*

(i) *for each pair of Fresnel kernels* $\mathscr{F}_1$, $\mathscr{F}_2$, *if* $\mathscr{F}_1 \circ \mathscr{F}$ *and* $\mathscr{F}_2 \circ \mathscr{F}$ *are defined then*

$$\mathscr{F}_1 \circ \mathscr{F} = \mathscr{F}_2 \circ \mathscr{F} \quad \textit{implies} \quad \mathscr{F}_1 = \mathscr{F}_2$$

(ii) *for each vector space V and any* $\mathscr{F}_1 \in FK(V, Y)$, $\mathscr{F}_1 \circ \mathscr{F}$ *is defined*

(iii) $\mathscr{F}^\dagger \circ \mathscr{F}$ *is defined*

(iv) $cl(\mathscr{F})$ *is an epimorphism*

(v) $\mathscr{F}_+(S(X)) \subset S(Y)$

(vi) $\mathscr{F}_+(S(X)) = S(Y)$

(vii) $\mathscr{F}_+(S(X)) \supset S(Y)$

(viii) $\mathscr{F}_-$ *is an injective mapping*

(ix) $(\mathscr{F}^\dagger)_+$ *is injective.*

Proof. Let $\rho = cl(\mathscr{F})$.

(ii) ⇒ (iii) is obvious;

(iii) ⇒ (iv). $\{0\} = \rho(\{0\}) \cap \rho(\{0\}) = \rho(\{0\})$;

(iv) ⇒ (ii) is obvious;

(i) ⇒ (iv). Assume that ρ is not an epimorphism. Then one can construct an isomorphism $\rho_1 \neq 1_{PhY}$ such that $\rho_1 \circ \rho = \rho$. Let $\mathscr{F}_0$ be such that $cl(\mathscr{F}_0) = \rho_1$. We have

$$cl(\mathscr{F}_0 \circ \mathscr{F}) = cl(\mathscr{F}_0) \circ cl(\mathscr{F}) = cl(\mathscr{F}),$$

hence $\mathscr{F}_0 \circ \mathscr{F} = \lambda \mathscr{F}$ for a certain $\lambda \in \mathbb{C}$. For $\mathscr{F}_1 = \mathscr{F}_0/\lambda$ we have then $\mathscr{F}_1 \circ \circ \mathscr{F} = I_V \circ \mathscr{F}$, but $\mathscr{F}_1 \neq I_V$.

(iv) $\Rightarrow$ (i). If $\mathscr{F}_1 \circ \mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}$ then $cl(\mathscr{F}_1) \circ cl(\mathscr{F}) = cl(\mathscr{F}_2) \circ cl(\mathscr{F})$, hence $cl(\mathscr{F}_1) = cl(\mathscr{F}_2)$ and therefore $\mathscr{F}_1 = \lambda \mathscr{F}_2$. But $\lambda \mathscr{F}_2 \circ \mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}$ implies $\lambda = 1$;

(iv) $\Rightarrow$ (vi) by Lemma 1;

(vi) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are obvious;

(vi) $\Rightarrow$ (viii) is easy;

(v) $\Rightarrow$ (iv), (vii) $\Rightarrow$ (iv) and (viii) $\Rightarrow$ (iv). Let us assume that ρ is not an epimorphism. Then $\rho(X)^{\S}$ contains non-zero elements. Let $y \oplus r \in Ph\, Y$ be such an element. From Lemma 3 (see Appendix) we have

$$\left(r - \frac{1}{i}\mathscr{L}_y\right)\mathscr{F} = 0.$$

From this equation we see that for any $\psi \in S(X)$

$$\left(r - \frac{1}{i}\mathscr{L}_y\right)(\mathscr{F}_+ \psi) = 0.$$

No nontrivial element $\mathscr{F}_+ \psi$ of $S(Y)$ satisfies this equation: this contradicts (iv) and (vi). The same equation implies that for each $\chi \in S(Y)$

$$\mathscr{F}_-\left(r + \frac{1}{i}\mathscr{L}_y\right)\chi = 0$$

and this contradicts (viii).

(viii) $\Longleftrightarrow$ (ix), since

(29) $$(\mathscr{F}^\dagger)_+ \chi = \overline{(\mathscr{F}_- \overline{\chi})}.$$ ∎

DEFINITION 9. A Fresnel kernel is called an *epimorphism* if it satisfies the conditions stated in Proposition 12.

One can easily state the «adjoint» version of Proposition 12, giving conditions for a Fresnel kernel $\mathscr{F}$ which are equivalent to

(iv†) $cl(\mathscr{F})$ is a monomorphism.

Such kernels are called *monomorphisms*. Then we introduce also *isomorphisms* - kernels bieng at the same time mono - and epi-morphisms. The following proposition characterizes isomorphisms.

PROPOSITION 13. *For $\mathcal{F} \in FK(Y, X)$ the following conditions are equivalent:*

(i) *$\mathcal{F}$ is an isomorphism*

(ii) *there exists $\lambda > 0$ such that $\mathcal{F}^\dagger \circ \mathcal{F} = \lambda I_X$ and $\mathcal{F} \circ \mathcal{F}^\dagger = I_Y$*

(iii) *there exists $\lambda > 0$ such that*

$$(\mathcal{F}_Y^X)^\dagger \mathcal{F}_Y^X = \lambda (I_X)_X^X \text{ and } \mathcal{F}_Y^X (\mathcal{F}_Y^X)^\dagger = \lambda (I_Y)_Y^Y$$

(that is to say, $\mathcal{F}_Y^X$ is proportional to a unitary operator).

Proof. (i) ⇒ (ii). We have

$$cl(\mathcal{F}^\dagger \circ \mathcal{F}) = cl(\mathcal{F})^\dagger \circ cl(\mathcal{F}) = 1_{Ph X} \text{ and } cl(\mathcal{F} \circ \mathcal{F}^\dagger) = 1_{Ph Y}. \tag{30}$$

Therefore $\mathcal{F}^\dagger \circ \mathcal{F} = \lambda I_X$ and $\mathcal{F} \circ \mathcal{F}^\dagger = \eta I_Y$. But $\eta = \lambda$, since

$$\eta \mathcal{F} = (\mathcal{F} \circ \mathcal{F}^\dagger) \circ \mathcal{F} = \mathcal{F} \circ (\mathcal{F}^\dagger \circ \mathcal{F}) = \lambda \quad .$$

The opereator $(\mathcal{F}_Y^X) \circ \mathcal{F}_Y^X = \lambda (I_X)_X^X$ is positive, hence λ is positive.
(ii) ⇒ (i). It is easy to show that (30) is satisfied and it follows that $cl(\mathcal{F})$ is an isomorphism.
(ii) ⟺ (iii) is obvious. ■

Epimorphisms, monomorphisms and isomorphisms form separately three true categories, homomorphic with the corresponding subcategories of *LSR*.

The following decomposition theorem holds:

PROPOSITION 14. *For each Fresnel kernel $\mathcal{F}$ there exist an epimorphism $\mathcal{F}_1$, an isomorphism $\mathcal{F}_2$ and a monomorphism $\mathcal{F}_3$ such that $\mathcal{F} = \mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_1$. If $\mathcal{F} \in FK(X, X)$ is self-adjoint then we can set $\mathcal{F}_3 = \mathcal{F}_1^\dagger$.*

Proof. Let $cl(\mathcal{F}) = r_2^\dagger \circ \rho_0 \circ r_1$ be the canonical decomposition of $cl(\mathcal{F})$. Let us identify the reduced symplectic spaces corresponding to r_1 and r_2 with certain phase spaces so that we are dealing only with linear phase relations. Let $\mathcal{F}_1$, $\tilde{\mathcal{F}}_2$, $\mathcal{F}_3$ be such that $cl(\mathcal{F}_1) = r_1$, $cl(\tilde{\mathcal{F}}_2) = \rho_0$ and $cl(\mathcal{F}_3) = r_2$. Then

$$\mathcal{F} = \lambda \, \mathcal{F}_3 \circ \tilde{\mathcal{F}}_2 \circ \mathcal{F}_1 = \mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_1,$$

where $\mathcal{F}_2 = \lambda \, \tilde{\mathcal{F}}_2$. If $\mathcal{F}$ is self-adjoint, we have $r_2 = r_1$. ■

DEFINITION 10. A Fresnel kernel $\Delta \in FK(X, X)$ is called *positive* if for each $\psi \in S(X)$ $\langle \psi \mid \Delta_X^X \psi \rangle \geqslant 0$.

PROPOSITION 15. *A Fresnel kernel Δ is positive if and only if there exists $\mathcal{F}$ such that $\Delta = \mathcal{F}^\dagger \circ \mathcal{F}$.*

Proof. If Δ is positive then it is also self-adjoint, by the standard reasoning. Consider a decomposition $\Delta = \mathscr{F}_1^\dagger \circ \mathscr{F}_0 \circ \mathscr{F}_1$ as in Proposition 14. Let V be such that $\mathscr{F}_0 \in FK(V, V)$. $(\mathscr{F}_0)_+$ is proportional to a unitary operator and it is a positive operator by the positiveness of Δ. It follows that $\mathscr{F}_0 = \lambda I_V$ with $\lambda > 0$. As $\mathscr{F}$ one can then take $\mathscr{F} = \sqrt{\lambda}\ \mathscr{F}_1$.

The sufficiency of the condition is obvious. ■

From Proposition 15 it follows that if Δ is a positive Fresnel kernel then $cl(\Delta)$ is positive.

d) *Quotient objects, subobjects and equivalence*

The following proposition characterizes equivalent quotient objects in our partial category (cf. [2]).

PROPOSITION 16. *Let* $\mathscr{F}_1 \in FK(Y_1, X)$, $\mathscr{F}_2 \in FK(Y_2, X)$ *be two epimorphisms. Then the following conditions are equivalent:*

(i) *there exists an isomorphism* $\mathscr{F} \in FK(Y_2, Y_1)$ *such that*

$$\mathscr{F}_2 = \mathscr{F} \circ \mathscr{F}_1$$

(ii) *there exists* $\lambda > 0$ *such that* $\mathscr{F}_2^\dagger \circ \mathscr{F}_2 = \lambda\, \mathscr{F}_1^\dagger \circ \mathscr{F}_1$

(iii) $\mathrm{Im}\,(\mathscr{F}_1^\dagger)_X^Y = \mathrm{Im}\,(\mathscr{F}_2^\dagger)_X^Y$

(iv) $\mathrm{Im}\,(\mathscr{F}_1)_X^Y = \mathrm{Im}\,(\mathscr{F}_2)_X^Y$

(v) $\ker\,(\mathscr{F}_1)_{Y_1}^X = \ker\,(\mathscr{F}_2)_{Y_2}^X$

(vi) $cl(\mathscr{F}_1)$ *and* $cl(\mathscr{F}_2)$ *are isomorphic quotient objects of* PhX.

Proof. (i) $\Rightarrow$ (ii). $\mathscr{F}_2^\dagger \circ \mathscr{F}_2 = (\mathscr{F}_1^\dagger \circ \mathscr{F}^\dagger) \circ (\mathscr{F} \circ \mathscr{F}_1) = \mathscr{F}_1^\dagger \circ \lambda I_{Y_1} \circ \mathscr{F}_1 = \lambda \mathscr{F}_1^\dagger \circ$ $\circ\, \mathscr{F}_1$

(ii) $\Rightarrow$ (iii). $\mathrm{Im}\,(\mathscr{F}_2^\dagger)_X^Y = \mathrm{Im}(\mathscr{F}_2^\dagger \circ \mathscr{F}_2)_X^X = \mathrm{Im}\,(\mathscr{F}_1^\dagger \circ \mathscr{F}_1)_X^X = \mathrm{Im}\,(\mathscr{F}_1)_X^Y$

(ii) $\Rightarrow$ (iv) by formula (29)

(iv) $\Rightarrow$ (v) because $\ker\,(\mathscr{F}_1)_{Y_1}^X = [\mathrm{Im}\,(\mathscr{F}_1)_X^Y]^0$

(v) $\Rightarrow$ (vi). For $j = 1, 2$ set $K_j = cl(\mathscr{F}_j)(Ph\, Y_j)$. From Lemma 3 (Appendix),

$$x \oplus p \in K_1^\S \Longleftrightarrow \left(p - \frac{1}{i}\mathscr{L}_x\right)\mathscr{F}_1^\dagger = 0.$$

It is easy to see that the latter equality is equivalent to

$$\left(p - \frac{1}{i}\mathscr{L}_x\right)S(X) \subset \ker\,(\mathscr{F}_1)_{Y_1}^X.$$

By the assumption this is equivalent to

$$\left(p - \frac{1}{i} \mathscr{L}_x\right) S(X) \subset \ker (\mathscr{F}_2)^X_{Y_2}$$

and also to

$$\left(p - \frac{1}{i} \mathscr{L}_x\right) \mathscr{F}_2^\dagger = 0.$$

The latter equality is equivalent to

$$x \oplus p \in K_2^\S$$

(by Lemma 3). It follows that $K_1 = K_2$.

(vi) ⇒ (i). There exists an isomorphism ρ such that $cl(\mathscr{F}_2) = \rho \circ cl(\mathscr{F}_1)$. Let $\mathscr{F}_0$ be such that $cl(\mathscr{F}_0) = \rho$. We have then $cl(\mathscr{F}_2) = cl(\mathscr{F}_0 \circ \mathscr{F}_1)$, hence $\mathscr{F}_2 = \lambda \mathscr{F}_0 \circ \mathscr{F}_1 = \mathscr{F} \circ \mathscr{F}_1$, where $\mathscr{F} = \lambda \mathscr{F}_0$. ■

It follows from Proposition 16 that the equivalence classes of isomorphic quotient objects are in one-to-one correspondence with positive Fresnel kernels modulo a multiplicative constant (point (ii) of the proposition). The description of subobjects is similar and can be obtained by passing to adjoint kernels.

6. UNITARITY CONDITION: A MODIFICATION OF THE PARTIAL CATEGORY

In the partial category considered so far, ismorphisms are kernels proportional to unitary kernels (Proposition 13). Now, let us consider a partial category obtained from the previous one by removing all isomorphisms which are not unitary kernels. We refer to this new partial category as to the *partial category of normalized Fresnel kernels*. Thus, isomorphisms in this modified partial category preserve scalar products - the structure naturally existing in spaces $S(X)$.

a) *Modified description of isomorphic quotient objects and subobjects*

In the partial category of normalized Fresnel kernels, equivalence classes of isomorphic quotient objects (also subobjects) are in one-to-one correspondence with positive Fresnel kernels (no arbitrary constants appear).

b) *Scalar product Schwartz spaces associated with positive kernels*

With any positive $\Delta \in FK(X, X)$ we associate the space

$$S(X, \Delta) = S(X)/\ker \Delta_+ .$$

This is a Fréchet space as the quotient of a Fréchet space by a closed subspace. There is a *scalar product* on $S(X, \Delta)$, defined by

$$\langle [\psi_1] | [\psi_2] \rangle_\Delta = \langle \psi_1 | \Delta_+ \psi_2 \rangle \text{ for } \psi_1, \psi_2 \in S(X),$$

where $[\psi]$ denotes the projection of $\psi \in S(X)$ in $S(X, \Delta)$. The non-degeneracy of the scalar product follows from Proposition 15:

$$0 = \langle \psi | \Delta_+ \psi \rangle = \langle \psi | (\mathscr{F}_+)^\dagger \mathscr{F}_+ \psi \rangle = \langle \mathscr{F}_+ \psi | \mathscr{F}_+ \psi \rangle$$

implies $\psi \in \ker \mathscr{F}_+ = \ker (\mathscr{F}_+)^\dagger \mathscr{F}_+ = \ker \Delta_+$. Here $\mathscr{F} \in FK(Y, X)$ is any quotient object belonging to the equivalence class distinguished by Δ, i.e. $\Delta = \mathscr{F}^\dagger \circ \mathscr{F}$. For any such $\mathscr{F}$ the continuous linear map $\mathscr{F}_+ : S(X) \to S(Y)$ defines a continuous bijection

$$\mathscr{F}_{+,\Delta} : S(X, \Delta) \longrightarrow S(Y) \tag{31}$$

which is a topological isomorphism, by The Inverse Map Theorem. This is why we call $S(X, \Delta)$ a Schwartz space. In addition, $\mathscr{F}_{+,\Delta}$ preserves the scalar product:

$$\langle \phi_1 | \phi_2 \rangle_\Delta = \langle \mathscr{F}_{+,\Delta}\phi_1 | \mathscr{F}_{+,\Delta}\phi_2 \rangle \text{ for } \phi_1, \phi_2 \in S(X, \Delta).$$

7. THE PARTIAL CATEGORY OF REDUCED SPACES

a) *Reduced spaces*

DEFINITION 11. If $\Delta \in FK(X, X)$ is positive then the pair $(S(X, \Delta), \langle \cdot | \cdot \rangle_\Delta)$ is called the *reduced space of* $S(X)$ *with respect to* Δ.

Note that for $\Delta = I_X$ the reduced space is identical with $S(X)$ (with the standard scalar product).

It is clear that a general reduction leads out of the «phase» framework. Still, we can use «phase charts» of $S(X, \Delta)$ as given by various possible $\mathscr{F}_{+,\Delta}$ in formula (31). The following proposition shows that the transition from a «chart» to another «chart» is given by a Fresnel kernel.

PROPOSITION 17. *Let* $\mathscr{F} \in FK(Y, X)$, $\tilde{\mathscr{F}} \in FK(\tilde{Y}, X)$ *be such epimorphisms that* $\mathscr{F}^\dagger \circ \mathscr{F} = \tilde{\mathscr{F}}^\dagger \circ \tilde{\mathscr{F}} = \Delta$. *Then there exists a unique unitary Fresnel kernel* $\mathring{\mathscr{F}} \in FK(\tilde{Y}, Y)$ *such that*

$$\tilde{\mathscr{F}}_{+,\Delta} [\mathscr{F}_{+,\Delta}]^{-1} = \mathring{\mathscr{F}}_+ = \tilde{\mathscr{F}}_+ [\Delta_+]^{-1} (\mathscr{F}^\dagger)_+$$

(*here* $[\Delta_+]^{-1} : \operatorname{Im} \Delta_+ \to S(X, \Delta)$).

Proof. By Proposition 16 there exists a unique isomorphism $\mathring{\mathscr{F}} \in FK(\tilde{Y}, Y)$ such that $\tilde{\mathscr{F}} = \mathring{\mathscr{F}} \circ \mathscr{F}$. The proposition follows from two equalities,

$$\{\tilde{\mathscr{F}}_{+,\Delta}[\mathscr{F}_{+,\Delta}]^{-1}\}\ \mathscr{F}_+ = \tilde{\mathscr{F}}_+$$

and

$$\{\mathscr{F}_+[\Delta_+]^{-1}(\mathscr{F}^\dagger)_+\}\ \mathscr{F}_+ = \tilde{\mathscr{F}}_+[\Delta_+]^{-1}\Delta_+ = \tilde{\mathscr{F}}_+. \qquad \blacksquare$$

The Fresnel kernel $\mathring{\mathscr{F}}$ defined in Proposition 17 will be denoted by

$$\tilde{\mathscr{F}} \circ_\Delta \mathscr{F}^\dagger. \tag{32}$$

For $\tilde{\mathscr{F}} = \mathscr{F} \in FK(Y, X)$ we have $\mathscr{F} \circ_\Delta \mathscr{F}^\dagger = I_Y$.

b) *Fresnel relations*

DEFINITION 12. *A Fresnel relation from* $S(X_1, \Delta_1)$ *to* $S(X_2, \Delta_2)$ is an equivalence class of triples $(\mathscr{F}_2, \mathscr{F}_1; \mathscr{F}_{21})$, where $\mathscr{F}_1$, $\mathscr{F}_2$ are epimorphisms such that $\mathscr{F}_1^\dagger \circ \mathscr{F}_1 = \Delta_1$, $\mathscr{F}_2^\dagger \circ \mathscr{F}_2 = \Delta_2$ and $\mathscr{F}_{21} \in FK(\operatorname{cod} \mathscr{F}_2, \operatorname{cod} \mathscr{F}_1)$, the equivalence relation $\sim$ being given by

$$(\mathscr{F}_2, \mathscr{F}_1; \mathscr{F}_{21}) \sim (\tilde{\mathscr{F}}_2, \tilde{\mathscr{F}}_1; \tilde{\mathscr{F}}_{21}) \text{ if and only if}$$

$$\tilde{\mathscr{F}}_{21} = (\tilde{\mathscr{F}}_2 \circ_{\Delta_2} \mathscr{F}_2^\dagger) \circ \mathscr{F}_{21} \circ (\mathscr{F}_1 \circ_{\Delta_1} \tilde{\mathscr{F}}_1^\dagger).$$

Here cod $\mathscr{F}$ denotes the *codomain* of $\mathscr{F} \in FK(Y, X)$ and is defined by

$$\operatorname{cod} \mathscr{F} = Y.$$

The composition of Fresnel relations is defined by the composition of the Fresnel kernels representing them (if they are composable):

$$[(\mathscr{F}_3, \mathscr{F}_2; \mathscr{F}_{32})]_\sim \circ [(\mathscr{F}_2, \mathscr{F}_1; \mathscr{F}_{21})]_\sim = [(\mathscr{F}_3, \mathscr{F}_1; \mathscr{F}_{32} \circ \mathscr{F}_{21})].$$

Reduced spaces with Fresnel relations and the composition defined above form a partial category. Further discussion of its properties follows the same pattern as in the case of the partial category of Fresnel kernels. For example, the adjoint is defined as follows

$$[(\mathscr{F}_2, \mathscr{F}_1; \mathscr{F}_{21})]_\sim^\dagger = [(\mathscr{F}_1, \mathscr{F}_2; \mathscr{F}_{21}^\dagger)]_\sim .$$

In the particular case when $\Delta_1 = I_{X_1}$, we have $S(X_1, \Delta_1) = S(X_1)$ and it suffices to use the «identity chart» $\Delta_1 = I_{X_1}$ in Definition 12. The same applies to the case when $\Delta_2 = I_{X_2}$. In particular, Fresnel relations from $S(X_1)$ to $S(X_2)$ are simply Fresnel kernels belonging to $FK(X_2, X_1)$.

c) *Reductions*

DEFINITION 13. *The reduction relation from* $S(X)$ *to* $S(X, \Delta)$ is the Fresnel relation

$$\mathscr{R}_\Delta = \{(\mathscr{F}, I_X; \mathscr{F}) \mid \mathscr{F}^\dagger \circ \mathscr{F} = \Delta\}.$$

This definition is correct, since for $\mathscr{F}^\dagger \circ \mathscr{F} = \Delta = \tilde{\mathscr{F}}^\dagger \circ \tilde{\mathscr{F}}$ we have

$$\tilde{\mathscr{F}} = (\tilde{\mathscr{F}} \circ_\Delta \mathscr{F}^\dagger) \circ \mathscr{F} = (\tilde{\mathscr{F}} \circ_\Delta \mathscr{F}^\dagger) \circ \mathscr{F} \circ I_X$$

and therefore $(\mathscr{F}, I_X; \mathscr{F}) \sim (\tilde{\mathscr{F}}, I_X; \tilde{\mathscr{F}})$.

8. THE CATEGORY OF ISOMORPHISMS OF REDUCED SPACES AS A GENERALIZED CATEGORY OF FRESNEL KERNELS

a) *Images and coimages in the partial category of Fresnel kernels*

DEFINITION 14. (cf. [5]) *A coimage of a Fresnel kernel* $\mathscr{F}$ is any epimorphism $\mathscr{F}_1$ such that:

1° $\mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}_1$ for a certain $\mathscr{F}_2$

and

2° for any decomposition $\mathscr{F} = \mathscr{F}'_2 \circ \mathscr{F}'_1$, where $\mathscr{F}'_1$ is an epimorphism, there exists a morphism $\mathscr{F}_0$ such that $\mathscr{F}_1 = \mathscr{F}_0 \circ \mathscr{F}'_1$.

PROPOSITION 18. *Each Fresnel kernel $\mathscr{F}$ has a coimage. An epimorphism $\mathscr{F}_1$ is a coimage of $\mathscr{F}$ if and only if there exists a monomorphism $\mathscr{F}_2$ such that $\mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}_1$.*

Proof. By Proposition 14 one can find an epimorphism $\mathscr{F}_1$ and a monomorphism $\mathscr{F}_2$ such that $\mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}_1$. We shall show that $\mathscr{F}_1$ is a coimage of $\mathscr{F}$. In fact, if $\mathscr{F}'_1$ is an epimorphism such that

$$\mathscr{F} = \mathscr{F}'_2 \circ \mathscr{F}'_1,$$

we have

$$\mathscr{F}'_2 = \mathscr{F}''_3 \circ \mathscr{F}''_2,$$

where $\mathscr{F}''_2$ is an epimorphism and $\mathscr{F}''_3$ is a monomorphism. Then from the equality $\mathscr{F} = \mathscr{F}''_3 \circ (\mathscr{F}''_2 \circ \mathscr{F}'_1) = \mathscr{F}_2 \circ \mathscr{F}_1$ it follows that $\ker(\mathscr{F}''_2 \circ \mathscr{F}'_1)_+ = \ker \mathscr{F}_{1+}$ (see (21) for notation). From Proposition 16 it follows that there exists $\mathscr{F}''$ such that $\mathscr{F}_1 = \mathscr{F}'' \circ (\mathscr{F}''_2 \circ \mathscr{F}'_1) = (\mathscr{F}'' \circ \mathscr{F}''_2) \circ \mathscr{F}'_1 = \mathscr{F}_0 \circ \mathscr{F}'_1$. Now if we assume that $\mathscr{F}'_1$ is also a coimage of $\mathscr{F}$ then $\mathscr{F}'_1 = \mathscr{F}' \circ \mathscr{F}_1$ for a certain $\mathscr{F}'$. Since $\mathscr{F}_1 = \mathscr{F}_0 \circ \mathscr{F}'_1 = (\mathscr{F}_0 \circ \mathscr{F}') \circ \mathscr{F}_1$, it follows that $\mathscr{F}_0 \circ \mathscr{F}' = I$, $\mathscr{F}_0$ is an isomorphism, $\mathscr{F}_{20} = \mathscr{F}_2 \circ \mathscr{F}_0$ is a monomorphism and $\mathscr{F} = \mathscr{F}_{20} \circ \mathscr{F}'_1$. ∎

It is easy to see that classes of coimages of Fresnel kernels coincide with classes of isomorphic quotient objects in the partial category of Fresnel kernels.

Similar treatment as above applies to *images*.

b) *Fresnel kernels and isomorphisms of reduced spaces*

To each Fresnel relation $\phi = [(\mathscr{F}_2, \mathscr{F}_1, \mathscr{F}_{21})]_\sim$ from $S(X_1, \Delta_1)$ to $S(X_2, \Delta_2)$ there corresponds a unique Fresnel kernel $\mathscr{F} \in FK(X_2, X_1)$ given by

$$\mathscr{F} = \mathscr{F}_2 \circ \mathscr{F}_{21} \circ \mathscr{F}_1 = \mathscr{R}^\dagger_{\Delta_2} \circ \phi \circ \mathscr{R}_{\Delta_1}.$$

If ϕ is an isomorphism (i.e. $\mathscr{F}_{21}$ is an isomorphism) then $\mathscr{F}$ has the same coimages as Δ_1 and the same images as Δ_2.

Conversely, for $\mathscr{F} \in FK(X_2, X_1)$, Δ_1 having the same coimages as $\mathscr{F}$ and Δ_2 having the same images as $\mathscr{F}$, there is a unique decomposition

$$\mathscr{F} = \mathscr{R}^\dagger_{\Delta_2} \circ \phi \circ \mathscr{R}_{\Delta_1},$$

where ϕ is an isomorphism.

c) *The generalized category of Fresnel kernels*

Isomorphisms of reduced spaces form a category. We shall describe this category in terms of Fresnel kernels. To this end we use the results of the preceding paragraph. Formally, we introduce a new category:

1° *objects* are reduced spaces,

2° *morphisms* from $S(X_1, \Delta_1)$ to $S(X_2, \Delta_2)$ are Fresnel kernels with coimages and images specified by Δ_1 and Δ_2, respectively,

3° the *composition* of two mrophisms $\mathscr{F}_I = \mathscr{R}^\dagger_{\Delta_2} \circ \phi_I \circ \mathscr{R}_{\Delta_1}$, $\mathscr{F}_{II} = \mathscr{R}^\dagger_{\Delta_3} \circ \circ\, \phi_{II} \circ \mathscr{R}^\dagger_{\Delta_2}$ is the Fresnel kernel $\mathscr{F} \in FK(X_3, X_1)$ defined by

$$\mathscr{F}_+ = \mathscr{F}_{II+} \Delta^{-1}_{2+} \mathscr{F}_{I+}$$

(notation as in (21)).

Let us check that the above formula corresponds to the composition of isomorphisms of reduced spaces. If $\mathscr{F}_I = \mathscr{F}^\dagger_2 \circ \mathscr{F}_{21} \circ \mathscr{F}_1$ and $\mathscr{F}_{II} = \mathscr{F}^\dagger_3 \circ \mathscr{F}_{32} \circ \mathscr{F}_2$, then

$$\begin{aligned}
\mathscr{F}_{II+} \Delta^{-1}_{2+} \mathscr{F}_{I+} &= \mathscr{F}^\dagger_{3+} \mathscr{F}_{32+} \mathscr{F}_{2+} \Delta^{-1}_{2+} \mathscr{F}_{2+} \mathscr{F}_{21+} \mathscr{F}_{1+} = \\
&= \mathscr{F}^\dagger_{3+} \mathscr{F}_{32+} (\mathscr{F}_2 \circ_{\Delta_2} \mathscr{F}^\dagger_2)_+ \mathscr{F}_{21+} \mathscr{F}_{1+} = \mathscr{F}^\dagger_{3+} \mathscr{F}_{32+} \mathscr{F}_{21+} \mathscr{F}_{1+} \\
&= (\mathscr{F}^\dagger_3 \circ \mathscr{F}_{32} \circ \mathscr{F}_{21} \circ \mathscr{F}_1)_+ = \mathscr{R}^\dagger_{\Delta_3} \circ (\phi_{II} \circ \phi_I) \circ \mathscr{R}^\dagger_{\Delta_1}.
\end{aligned}$$

The described composition generalizes the operation in formula (32). Hence, we denote it by the same symbol

$$\mathscr{F}_{II} \circ_{\Delta_2} \mathscr{F}_I.$$

Let us note that each Fresnel kernel is represented by a morphism in this category. However, the representation is not unique (one can multiply Δ_1 and Δ_2 by positive numbers). Therefore reduced spaces with Fresnel kernels as morphisms form what is called a *generalized catetory* (see [6]), because of non--uniqueness of the domain and codomain.

9. APPENDIX

PROPOSITION 19. *Let $\mathscr{K}$ be a Frenel distribution on a vector space V. Let $\mathscr{K}$ be of the form* (20). *Let L be the Lagrangian subspace of PhV generated by Q. Then*

$$v \oplus w \in L \text{ if and only if } \left(w - \frac{1}{i}\mathscr{L}_v\right)\mathscr{K} = 0.$$

(here $\mathscr{L}_v$ denotes the derivative with respect to v).

Proof. ($\Rightarrow$). Since $w|_C = \mathrm{d}Q(v)$, we have $\left\langle \psi, \left(w - \frac{1}{i}\mathscr{L}_v\right)\mathscr{K}\right\rangle =$ $= \int_C \nu \circ \left(\left(w + \frac{1}{i}\mathscr{L}_v\right)\psi|_C\right) e^{iQ} = \int_C \nu \circ (\psi|_C)\left(w - \frac{1}{i}\mathscr{L}_v\right) e^{iQ} = 0$ for $\psi \in S(V)$
($\Leftarrow$). Suppose $\left(w - \frac{1}{i}\mathscr{L}_v\right)\mathscr{K} = 0$. Then for any $x \oplus p \in L$,

$$0 = \left[w - \frac{1}{i}\mathscr{L}_v, p - \frac{1}{i}\mathscr{L}_x\right]\mathscr{K} = \frac{1}{i}\,\omega_V(v + w, x + p)\mathscr{K}.$$

It follows that $v \oplus w \in L^{\S} = L$. ■

LEMMA 3. *If $\mathscr{F} \in FK(Y, X)$, $\rho = cl(\mathscr{F})$, $K = \rho(X)$, then $y \oplus r \in K^{\S}$ if and only if* $\left(r - \frac{1}{i}\mathscr{L}_y\right)\mathscr{F} = 0$.

Proof. Both conditions are equivalent to

$$y \oplus r \oplus 0 \oplus 0 \in \text{Graph}\,\rho.$$ ■

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